Lecture XXV: § 5.4 Linear Independence, Bases

Recall Spanning sets, linear independence, Bases:

Fix V a vector space & a set of "vectors" $S = \{\vec{v}_1, \ldots, \vec{v}_p\} \subset V$

- We define $Sp(S) = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_p\vec{v}_p : a_1, a_2, \ldots, a_p \text{ arbitrary}\}$

  It is the subspace of V spanned by S (S spans $Sp(S)$)

- We say S is linearly independent if the only solution to the dependency relation: $a_1\vec{v}_1 + \cdots + a_p\vec{v}_p = \underbrace{\vec{0}}_{\text{Neutral El. in } V}$ is $a_1 = a_2 = \cdots = a_p = 0$.

Examples:

	$\mathbb{R}^n$	$Mat_{m\times n}$	$\mathcal{F}$	$\mathcal{P}_n$	$C[0,1]$
$\vec{0}$	$\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$	zero matrix	constant 0 function	zero polynomial	constant 0 function

Definition: Fix a subspace W of V & $B = \{\vec{w}_1, \ldots, \vec{w}_k\}$ a subset of W.
Then B is a basis for W if (1) $W = Sp(B)$ ("B spans W")
& (2) B is linearly independent

§1 Examples:

① $Mat_{2\times 3} = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \ a_{11}, \ldots, a_{23} \text{ free} \right\}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = a_{11} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=E_{11}} + a_{12} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=E_{12}} + a_{13} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{=E_{13}} + a_{21} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{=E_{21}}$$
$$+ a_{22} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{=E_{22}} + a_{23} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=E_{23}}$$

E_{ij} = is a matrix with a 1 in (i,j)-entry and 0's everywhere else.
(size is understood from context.

- These 6 matrices span $Mat_{2\times 3}$

- They are li: A dependency relation looks like:

$$a_{11}E_{11} + a_{12}E_{12} + a_{13}E_{13} + a_{21}E_{21} + a_{22}E_{22} + a_{23}E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The calculation above rewrites the (LHS) as a matrix, we get:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The equality of matrices gives equality of entries on each side, so

$$a_{11} = a_{12} = a_{13} = a_{21} = a_{22} = a_{23} = 0.$$

<u>In general</u>: Basis for $Mat_{m\times n} = \{E_{11}, E_{12}, \ldots, E_{1n}, E_{21}, \ldots, E_{mn}\}$ ($m \cdot n$ elements)

<u>Obs</u>: This is how you find basis for subspaces of $Mat_{m\times n}$.

(2) Symmetric 3×3 matrices $= Sp(E_{11}, E_{12}+E_{21}, E_{13}+E_{31}, E_{22}, E_{23}+E_{32}, E_{33})$
$= W$ $\qquad$ last time

Check: These 6 matrices are a basis for W

(3) Decide if $\left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 5 \\ 0 & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$ is li/ld.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = a\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + b\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} + c\begin{bmatrix} 3 & 0 & 5 \\ 0 & 4 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a+b+3c & 0 & a+2b+5c+d \\ 0 & a+b+4c & 0 \end{bmatrix}$$

$\rightsquigarrow$ We get 3 equations

$$\begin{cases} a+b+3c = 0 \\ a+2b+5c+d = 0 \\ a+b+4c = 0 \end{cases}$$

3 equations & 4 unknowns $\rightsquigarrow$ solution is <u>NOT</u> unique, so <u>l.d</u>

<u>Solution</u>: $\left[\begin{array}{cccc|c} 1 & 1 & 3 & 0 & 0 \\ 1 & 2 & 5 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & 3 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow[R_1 \to R_1 - 3R_3]{R_2 \to R_2 - 2R_3} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$

$\xrightarrow[R_1 \to R_1 - R_2]{} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$ $\qquad$ $\begin{aligned} a - d &= 0 \\ b + d &= 0 \\ c &= 0 \end{aligned}$ $\qquad$ $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = d\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

$\uparrow \uparrow \uparrow$
a, b, c dep $\qquad\qquad$ $\hookrightarrow$ relation!

Relation:
$$\overset{A}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} - \overset{B}{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}} + \overset{D}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\{A, B, C, D\}$ is <u>not</u> li but we can drop any of A, B or D & get a li set spanning $Sp(A,B,C,D) = W$

↳ no other relation among A, B, C, D.

<u>Conclusion</u> $\{A, B, C\}$, $\{A, C, D\}$ & $\{B, C, D\}$ are bases for W. (This is the same algorithm we had for building bases for subspaces of $\mathbb{R}^n$ starting from a spanning set)

(3) $\mathcal{P}_2 = \{a_0 \cdot 1 + a_1 x + a_2 x^2\}$ has basis $\{1, x, x^2\}$

- Spanning is clear.
- LI? Use derivation + evaluation at $x=0$ repeatedly

$$0 = a_0 + a_1 x + a_2 x^2 \underset{x=0}{\rightsquigarrow} 0 = a_1$$

$$0 = 0' = a_1 + 2a_2 x \underset{x=0}{\rightsquigarrow} 0 = a_1$$

$$0 = 0'' = 2a_2 \underset{x=0}{\rightsquigarrow} 0 = 2a_2 \text{ so } a_2 = 0.$$

<u>In general</u>: $\mathcal{P}_n$ has basis $\{1, x, x^2, \ldots, x^n\}$ $(n+1)$ elements!

(4) Decide if $\{1, (x+1)^2, (x-1)^2, x^2\}$ is li / ld

Write $0 = a + b(x+1)^2 + c(x-1)^2 + dx^2$

<u>Option 1</u>: Take successive derivatives and evaluate at $x=0$. You MUST CHECK the answer

$$0 = a + b(x+1)^2 + c(x-1)^2 + dx^2 \underset{x=0}{\rightsquigarrow} 0 = a+b+c$$

$$0 = 0' = 2b(x+1) + 2c(x-1) + 2dx \underset{x=0}{\rightsquigarrow} 0 = 2b - 2c$$

$$0 = 0'' = 2b + 2c + 2d \underset{x=0}{\rightsquigarrow} 0 = 2b + 2c + 2d$$

Linear System

3 eqns
4 unknowns

$$a + b + c = 0 \rightsquigarrow a + 2b = 0 \quad a = -2b$$
$$2b - 2c = 0 \rightsquigarrow b = c$$
$$2b + 2c + 2d = 0 \rightsquigarrow 4b + 2d = 0 \quad d = -2b$$

Solution : $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = b \begin{bmatrix} -2 \\ 1 \\ 1 \\ -2 \end{bmatrix} \rightsquigarrow$ check if $0 = -2 + (x+1)^2 + (x-1)^2 - 2x^2$ ✓

So L.D.

Option 2: Evaluate at 4 convenient (random) values of x to get a linear system in a, b, c, d & solve. You MUST check your answer.

At $x=0$: $0 = a + b + c$

At $x=1$: $0 = a + 4b + d$

At $x=-1$: $0 = a + 4c + d$

At $x=2$: $0 = a + 9b + c + 4d$

Solution is $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = b \begin{bmatrix} -2 \\ 1 \\ 1 \\ -2 \end{bmatrix}$

Option 3: Expand & write the (RHS) as a polynomial & set coefficients to 0

$$\begin{aligned} 0 &= a + b(x+1)^2 + c(x-1)^2 + dx^2 \\ &= a + b(x^2+2x+1) + c(x^2-2x+1) + dx^2 \\ &= (a+b+c) + (2b-2c)x + (b+c+d)x^2 \end{aligned}$$

System: $\begin{cases} a+b+c=0 \\ 2b-2c=0 \\ b+c+d=0 \end{cases}$ Solution is $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = b \begin{bmatrix} -2 \\ 1 \\ 1 \\ -2 \end{bmatrix}$

Relation: $0 = -2 \cdot 1 + (x+1)^2 + (x-1)^2 + (-2)x^2$ is the only relation among the 4 polynomials, so removing any of these 4 gives a linearly independent set.

§2 Properties of Bases:

⚠ Not every vector space has a finite basis

For example, $V = \mathcal{F}$ = functions of one variable

Last time, we saw that $\{1, x, x^2, x^3, \dots, x^n\}$ is linearly independent for any $n \geq 0$. So we can find arbitrarily large sets of lin. indep. vectors in V. This means V cannot possibly have a finite basis.

	$\mathbb{R}^n$	$Mat_{m\times n}$	$\mathbb{F}$	$\mathbb{P}_n$	$C[0,1]$
Basis	$\{\vec{e}_1, \dots, \vec{e}_n\}$ n elem.	$\{E_{11}, \dots, E_{mn}\}$ m·n elem	none finite	$\{1, x, \dots, x^n\}$ (n+1) elem	

Fix V a vector space and $B = \{\vec{v}_1, \dots, \vec{v}_p\}$ a basis for V

1. If $S = \{\vec{w}_1, \dots, \vec{w}_q\}$ is in V and $q > p$, then S is linearly dependent.

<u>Proof</u> Write $\vec{w}_1, \dots, \vec{w}_q$ using the basis B.

$$\begin{cases} \vec{w}_1 = a_{11}\vec{v}_1 + a_{21}\vec{v}_2 + \dots + a_{p1}\vec{v}_p \\ \vec{w}_2 = a_{12}\vec{v}_1 + a_{22}\vec{v}_2 + \dots + a_{p2}\vec{v}_p \\ \vdots \\ \vec{w}_q = a_{1q}\vec{v}_1 + a_{2q}\vec{v}_2 + \dots + a_{pq}\vec{v}_p \end{cases} \rightsquigarrow A = (a_{ij}) \text{ } p\times q \text{ matrix}$$

A linear expression $x_1\vec{w}_1 + \dots + x_q\vec{w}_q = \vec{0}$ is the same as a homogeneous system:

$$\underset{p\times q}{A} \underset{q\times 1}{\begin{bmatrix} x_1 \\ \vdots \\ x_q \end{bmatrix}} = \underset{p\times 1}{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}$$

equations $= p$
unknowns $= q$ $\quad q > p$

Hence, nontrivial solutions exist & S is lin dep.

2. If $\tilde{B} = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_q\}$ is another basis for V, then $q = p$.

<u>Proof</u> Same as for subspaces of $\mathbb{R}^n$. If $q > p$ then $\tilde{B}$ is lin. dep by 1, but this cannot be the case because $\tilde{B}$ is a basis

If $p > q$, then using 1 for the basis $\tilde{B}$ says B is linearly dep, which again can't happen. Conclusion is $p = q$.

Consequence: $\dim(V)$ = dimension of V = size of any basis for V.

This is a well-defined non-negative integer.

3 We have coordinates of vectors in V relative to bases (Next Time!)