Lecture XXVI: §5.4 Coordinates relative to Bases
§5.7 Linear Transformations

Last time: V vector space, $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ is a basis for V if

(1) $V = Sp(\mathcal{B}) = \{a_1\vec{v}_1 + \cdots + a_p\vec{v}_p : a_1, \ldots, a_p \text{ arbitrary}\}$ ($\mathcal{B}$ spans V)

& (2) $\mathcal{B}$ is l.i. (only solution to $\vec{0} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_p\vec{v}_p$ in $(a_1, \ldots, a_p)$ is $a_1 = a_2 = \cdots = a_p = 0$

Examples:

	$\mathbb{R}^n$	$Mat_{m\times n}$	$\mathcal{P}_n$	$\mathbb{F}$	$C[0,1]$
$\vec{0}$	$\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$	$\begin{bmatrix}0 & \cdots & 0\\ 0 & \cdots & 0\end{bmatrix}$	zero poly	constant 0 function	constant 0 function
Basis	$\{\vec{e}_1, \ldots, \vec{e}_n\}$ n elem	$\{E_{11}, \ldots, E_{mn}\}$ $m\cdot n$ elem	$\{1, x, \ldots, x^n\}$ $(n+1)$ elem	none finite	none finite
dim	n	$m\cdot n$	$n+1$	∞	∞

$E_{ij} :=$ $m\times n$ matrix with 1 in (i,j) entry & 0's elsewhere.

Obs: $\{\vec{0}\}$ is a vector space with No basis. ($\dim(\{\vec{0}\}) = 0$)

Algorithm: S spanning set $\longrightarrow$ $\mathcal{B}$ basis

How? Remove 1 element at a time from S using dependency relations in S until what remains is l.i. (this is the output $\mathcal{B}$)

§1. Properties of bases:

Fix V a vector space and $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ a basis for V

1. If $S = \{\vec{w}_1, \ldots, \vec{w}_q\}$ is in V and $q > p$, then S is linearly dependent.

Proof Write $\vec{w}_1, \ldots, \vec{w}_q$ using the basis $\mathcal{B}$.

$$\begin{cases} \vec{w}_1 = a_{11}\vec{v}_1 + a_{21}\vec{v}_2 + \cdots + a_{p1}\vec{v}_p \\ \vec{w}_2 = a_{12}\vec{v}_1 + a_{22}\vec{v}_2 + \cdots + a_{p2}\vec{v}_p \\ \vdots \\ \vec{w}_q = a_{1q}\vec{v}_1 + a_{2q}\vec{v}_2 + \cdots + a_{pq}\vec{v}_p \end{cases} \rightsquigarrow A = (a_{ij}) \quad p\times q \text{ matrix}$$

A linear expression $x_1\vec{w}_1 + \cdots + x_q\vec{w}_q = \vec{0}$ is the same as a homogeneous system:

$$\underset{p\times q}{A}\ \underset{q\times 1}{\begin{bmatrix}x_1\\ \vdots \\ x_q\end{bmatrix}} = \underset{p\times 1}{\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}}$$

equations $= p$
unknowns $= q$
$q > p$

Hence, non trivial solutions exist & S is lin dep.

② If $\tilde{B} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_q\}$ is another basis for V, then $q = p$.

Proof Same as for subspaces of $\mathbb{R}^n$. If $q > p$ then $\tilde{B}$ is lin. dep by ①, but this cannot be the case because $\tilde{B}$ is a basis.

If $p > q$, then using ① for the basis $\tilde{B}$ says B is linearly dep, which again can't happen. Conclusion is $p = q$.

Consequence: $\dim(V)$ = dimension of V = size of any basis for V.

This is a well-defined non-negative integer.

③ For every $\vec{v}$ in V, there are uniquely determined scalars $a_1, \ldots, a_p$ so that

$$\vec{v} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_p \vec{v}_p$$

Write $[\vec{v}]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}$ vector in $\mathbb{R}^p$ = coordinates of $\vec{v}$ relative to the basis B.

Proof: Since $Sp(B) = V$, the $a_1, a_2, \ldots, a_p$ must exist. Uniqueness follows from l.i. of B: 2 different solutions will give a dependency relation among $\vec{v}_1, \ldots, \vec{v}_p$.

Examples: ① $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + d = 0 \right\}$

$$= \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \; a, b, c \text{ free} \right\} = Sp\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right)$$

basis. B

$$\left[\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \right]_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ vector in } \mathbb{R}^3$$

We often view this operation of writing coordinates relative to a basis as a "linear transformation"

$$\begin{array}{ccc} V & \longrightarrow & \mathbb{R}^p \\ \vec{v} & \longmapsto & [\vec{v}]_B \end{array} \qquad p = \dim V$$

Note for $\mathcal{B}_1 = \left\{\begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}1&0\\0&-1\end{bmatrix}\right\}$, then $\left[\begin{bmatrix}a&b\\c&-a\end{bmatrix}\right]_{\mathcal{B}_1} = \begin{bmatrix}b\\c\\a\end{bmatrix}$

(order of vectors in the basis matters when computing coordinates!)

(2) Let $V = \mathrm{Mat}_{2\times 2}$ & $\mathcal{B} = \left\{A_1 = \begin{bmatrix}1&0\\0&0\end{bmatrix}, A_2 = \begin{bmatrix}1&0\\1&0\end{bmatrix}, A_3 = \begin{bmatrix}0&0\\0&1\end{bmatrix}, A_4 = \begin{bmatrix}0&1\\0&1\end{bmatrix}\right\}$

Verify that $\mathcal{B}$ is a basis for V & write the coordinates of $\begin{bmatrix}2&-1\\-1&2\end{bmatrix}$ relative to $\mathcal{B}$

Solution: A typical element of V is $\begin{bmatrix}a&b\\c&d\end{bmatrix} = a\,E_{11} + b\,E_{12} + c\,E_{21} + d\,E_{22}$

We want to rewrite it in terms of A_1, A_2, A_3, A_4.

Option 1: Write the equation in x, y, z & w & solve it:

$$(*) \quad \begin{bmatrix}a&b\\c&d\end{bmatrix} = x\begin{bmatrix}1&0\\0&0\end{bmatrix} + y\begin{bmatrix}1&0\\1&0\end{bmatrix} + z\begin{bmatrix}0&0\\0&1\end{bmatrix} + w\begin{bmatrix}0&1\\0&1\end{bmatrix}$$
$$= \begin{bmatrix}x+y & w\\ y & z+w\end{bmatrix}$$

We get 4 linear equations by equating entries

$$\begin{cases} x+y = a & \rightsquigarrow x + c = a, \text{ so } x = a-c\\ w = b & \\ y = c & \\ z+w = d & \rightsquigarrow z + b = d, \text{ so } z = d-b\end{cases}$$

We get a unique solution for (*) no matter what a, b, c, d are, so $\mathcal{B}$ is linearly indep (uniqueness part) & spans (because solutions exist).

Conclude: $\mathcal{B}$ is a basis.

$$\left[\begin{bmatrix}a&b\\c&d\end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix}x\\y\\z\\w\end{bmatrix} = \begin{bmatrix}a-c\\c\\d-b\\b\end{bmatrix}$$

In particular, for $a = 2 = d$, $b = c = -1$,

$$\left[\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix}2-(-1)\\-1\\2-(-1)\\-1\end{bmatrix} = \begin{bmatrix}3\\-1\\3\\-1\end{bmatrix} \text{ in } \mathbb{R}^4$$

Option 2: Show $E_{11}, E_{12}, E_{13}, E_{14}$ in $\mathrm{Sp}(\mathcal{B})$ using (*)

$E_{11} = A_1$, $E_{22} = A_3$

- $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x+y & w \\ y & z+w \end{bmatrix}$ $\quad w=1, \; y=0, \; x=0, \; z=-1$

$E_{12} = -A_3 + A_4$

- $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} x+y & w \\ y & z+w \end{bmatrix}$ $\quad w=0 \quad y=1 \quad x=-1, \; z=0$

$E_{21} = -A_1 + A_2$

Solutions are unique, so $\mathcal{B}$ is li

- To find the coordinates we write E_{ij} using A_1, through A_4

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 2E_{11} - E_{12} - E_{21} + 2E_{13}$$
$$= 2A_1 - (-A_3 + A_4) - (-A_1 + A_2) + 2A_3$$
$$= \underline{(2+1)} A_1 + \underline{(-1)} A_2 + \underline{(2+1)} A_3 \underline{-} A_4$$

these 4 numbers give the coordinates relative to B

$$\left[\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right]_B = \begin{bmatrix} 3 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

Next Time: Give V with bases $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_p\}$ we can check li, spanning of $S = \{\vec{w}_1, \dots, \vec{w}_k\}$ by working in $\mathbb{R}^p$.

(1) S is l. indep in V if, and only if, $\{[\vec{w}_1]_B, \dots, [\vec{w}_k]_B\}$ is li in $\mathbb{R}^p$.

(2) S spans V if, and only if, $\{[\vec{w}_1]_B, \dots, [\vec{w}_k]_B\}$ spans $\mathbb{R}^p$.

Consequence: Fix $\mathbb{V} \neq \{\vec{0}\}$ a vector space with $\dim \mathbb{V} = p$. Then:

(1) Any set of $p+1$ or more vectors in $\mathbb{V}$ is lin. dep.

(2) ______ $p-1$ ______ cannot span $\mathbb{V}$

(3) ______ p lin. indep vectors in V is a basis for V.

(4) ______ p vectors that spans V is a basis for V.

Back to our example: Take $V = \mathrm{Mat}_{2\times 2}$ $B_1 = \{E_{11}, E_{12}, E_{21}, E_{22}\}$

$$[A_1]_{B_1} = \left[\begin{bmatrix}1&0\\0&0\end{bmatrix}\right]_{B_1} = [E_{11}]_{B_1} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$[A_2]_{B_1} = \left[\begin{bmatrix}1&0\\1&0\end{bmatrix}\right]_{B_2} = [E_{11}+E_{21}]_{B_2} = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

$$[A_3]_{B_1} = \left[\begin{bmatrix}0&0\\0&1\end{bmatrix}\right]_{B_2} = [E_{22}]_{B_2} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$[A_4]_{B_1} = \left[\begin{bmatrix}0&1\\0&1\end{bmatrix}\right]_{B_2} = [E_{12}+E_{22}]_{B_2} = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$$

Then $\{A_1, A_2, A_3, A_4\}$ is a basis for $\mathrm{Mat}_{2\times 2}$ because $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^4$ (it's easy to check linear independence & the size 4 agrees with $\dim(\mathbb{R}^4)$)

§2. Linear Transformations:

The construction extend that of a linear transf $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Definition: Given two vector spaces V & W and a function

$$T: V \longrightarrow W \qquad \vec{v} \longmapsto T(\vec{v}) \text{ (vector in } W)$$

we say T is a linear transformation if

(1) $T(\underbrace{\vec{u}+\vec{v}}_{\text{in } V}) = T(\vec{u}) \underbrace{+}_{\text{in } W} T(\vec{v})$ for any 2 vectors $\vec{u}, \vec{v}$ in V

(2) $T(\underbrace{c\cdot\vec{v}}_{\text{in } V}) = c \underbrace{\cdot}_{\text{in } W} T(\vec{v})$ for any $\vec{v}$ in V & c any scalar.

In short: T respects addition & scalar multiplication, ie the operations defining vector spaces

Remark $T(\vec{0}_V) = \vec{0}_W$ if T is linear

(Proof: take $c=0$ in (2) & use $0\cdot\vec{v} = \vec{0}_V$, $0\cdot T(\vec{v}) = \vec{0}_W$.

Examples: ① $T: \frac{d}{dx}: \mathcal{P}_3 \longrightarrow \mathcal{P}_2$ is a linear transformation

(1) $(f+g)'_{(x)} = f'_{(x)} + g'_{(x)}$ (2) $(c f_{(x)})' = c f'_{(x)}$

Explicitly $T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = a_1 + 2a_2 x + 3a_3 x^2$

Note: At the level of coordinates (with respect to standard basis for $\mathcal{P}_3$ & $\mathcal{P}_2$) : $\underset{\text{in } \mathcal{P}_3}{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}} \longrightarrow \underset{\text{in } \mathcal{P}_2 \checkmark}{\begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \end{bmatrix}}$ is a linear transf $\mathbb{R}^4 \to \mathbb{R}^3$

(2) $V = W = \mathcal{P}_3$ & $T: \mathcal{P}_3 \longrightarrow \mathcal{P}_3$ given by $T(f)_{(x)} = f_{(x+1)}$

T is a linear transformation

(1) $T(f+g)_{(x)} = (f+g)_{(x+1)} = f_{(x+1)} + g_{(x+1)} = T(f)_{(x)} + T(g)_{(x)}$

(2) $T(cf)_{(x)} = (cf)_{(x+1)} = c f_{(x+1)} = c\, T(f)_{(x)}$.

Explicitly : $T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3$

$= a_0 + a_1(x+1) + a_2(x^2+2x+1) + a_3(x^3+3x^2+3x+1)$

$= (a_0+a_1+a_2+a_3) + (a_1+2a_2+3a_3)x + (a_2+3a_3)x^2 + a_3 x^3$

Note: At the level of coordinates (with respect to standard basis for $\mathcal{P}_3$) : $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \longrightarrow \begin{bmatrix} a_0+a_1+a_2+a_3 \\ a_1+2a_2+3a_3 \\ a_2+3a_3 \\ a_3 \end{bmatrix}$

This is a linear Transf $\mathbb{R}^4 \longrightarrow \mathbb{R}^4$