Lecture XXVII: §5.7 Linear Transformations

§.1. Linear Transformations:

The construction extend that of a linear transf $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Definition: Given two vector spaces V & W and a function

$$T: V \longrightarrow W \qquad \vec{v} \longmapsto T(\vec{v}) \text{ (vector in } W)$$

we say T is a <u>linear transformation</u> if

(1) $T(\underbrace{\vec{u}+\vec{v}}_{\text{in } V}) = T(\vec{u}) \underbrace{+}_{\text{in } W} T(\vec{v})$ for any 2 vectors $\vec{u}, \vec{v}$ in V

(2) $T(\underbrace{c\cdot\vec{v}}_{\text{in } V}) = c \underbrace{\cdot}_{\text{in } W} T(\vec{v})$ for any $\vec{v}$ in V & c any scalar.

In short: T respects addition & scalar multiplication, ie the operations defining vector spaces.

Remark $T(\vec{0}_V) = \vec{0}_W$ if T is linear

(Proof: take $c=0$ in (2) & use $0\cdot\vec{v} = \vec{0}_V$, $0\cdot T(\vec{v}) = \vec{0}_W$.

Examples: ① $T: \frac{d}{dx}: \mathcal{P}_3 \longrightarrow \mathcal{P}_2$ is a linear transformation

(1) $(f+g)'_{(x)} = f'_{(x)} + g'_{(x)}$ (2) $(c f_{(x)})' = c f'_{(x)}$

Explicitly $T(\underbrace{a_0 + a_1 x + a_2 x^2 + a_3 x^3}_{\text{in } \mathcal{P}_3}) = \underbrace{a_1 + 2a_2 x + 3a_3 x^2}_{\text{in } \mathcal{P}_2 \checkmark}$

Note: At the level of coordinates (with respect to standard basis for $\mathcal{P}_3$ & $\mathcal{P}_2$): $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \longrightarrow \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \end{bmatrix}$ is a linear transf $\mathbb{R}^4 \to \mathbb{R}^3$

② $V = W = \mathcal{P}_3$ & $T: \mathcal{P}_3 \longrightarrow \mathcal{P}_3$ given by $T(f)_{(x)} = f_{(x+1)}$

T is a linear transformation

(1) $T(f+g)_{(x)} = (f+g)_{(x+1)} = f_{(x+1)} + g_{(x+1)} = T(f)_{(x)} + T(g)_{(x)}$

(2) $T(cf)_{(x)} = (cf)_{(x+1)} = c\, f_{(x+1)} = c\, T(f)_{(x)}$.

Explicitly: $T(a_0+a_1x+a_2x^2+a_3x^3) = a_0+a_1(x+1)+a_2(x+1)^2+a_3(x+1)^3$

$$= a_0+a_1(x+1)+a_2(x^2+2x+1)+a_3(x^3+3x^2+3x+1)$$

$$= (a_0+a_1+a_2+a_3)+(a_1+2a_2+3a_3)x+(a_2+3a_3)x^2+a_3x^3$$

Note: At the level of coordinates (with respect to standard basis for P_3):

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \longrightarrow \begin{bmatrix} a_0+a_1+a_2+a_3 \\ a_1+2a_2+3a_3 \\ a_2+3a_3 \\ a_3 \end{bmatrix}$$

This is a linear Transf $\mathbb{R}^4 \longrightarrow \mathbb{R}^4$

(3) $T: P_3 \longrightarrow P_4$ $\quad f_{(x)} \longmapsto \int_0^x f(t)\,dt$ is linear.

Explicitly $T(a_0+a_1x+a_2x^2+a_3x^3) = \int_0^x a_0+a_1t+a_2t^2+a_3t^3\,dt$

$$= a_0t+a_1\frac{t^2}{2}+a_2\frac{t^3}{3}+a_3\frac{t^4}{4}\Big|_0^x$$

$$= a_0x+\frac{a_1}{2}t^2+\frac{a_2}{3}t^3+\frac{a_3}{4}t^4$$

In coordinates $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \longmapsto \begin{bmatrix} 0 \\ a_0 \\ a_1/2 \\ a_2/3 \\ a_3/4 \end{bmatrix}$ is a linear transf $\mathbb{R}^4 \longrightarrow \mathbb{R}^5$

(4) $V=C[0,1]$ continuous function on $0 \le x \le 1$

$T: V \longrightarrow \mathbb{R}$ $\quad T(f) = \int_0^1 f(x)\,dx$ (= area under the graph of f)

T is a linear transformation by properties of integration.

(5) $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\quad T(\vec{x}) = A\vec{x}$ for a fixed $m \times n$ matrix A

is a linear transformation.

§2. Nullspace & Range: $T: V \longrightarrow W$ linear Transf. between vector spaces.

Def: The nullspace $\mathcal{N}(T)$ of T is the subspace of V defined by

$$\mathcal{N}(T) = \{\vec{v} \text{ in } V : T(\vec{v}) = \vec{0}_W\}$$

(It's a subspace because $\underset{(S1)}{T(\vec{0}_V) = \vec{0}_W}$, $\underset{(S2)}{\vec{0}_W + \vec{0}_W = \vec{0}_W}$ & $\underset{(S3)}{c \cdot \vec{0}_W = \vec{0}_W}$)

Def: The range $\mathcal{R}(T)$ is the subspace of W defined by

$$\mathcal{R}(T) = \{\vec{w} \text{ in } W \;:\; \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \text{ in } V\}$$

Def: $\text{nullity}(T) = \dim(\mathcal{N}(T))$, $\quad \text{rank}(T) = \dim(\mathcal{R}(T))$

Examples: ① $T: M_{2\times 3} \longrightarrow \mathcal{P}_5$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \longrightarrow a + bx + cx^2 + dx^3 + ex^4 + fx^5$$

- T is linear
- $\mathcal{N}(T) = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}$ because there is only 1 way to write the zero polynomial in $\mathcal{P}_5$, namely as $0 + 0\cdot x + 0x^2 + 0x^3 + 0x^4 + 0x^5$
- $\mathcal{R}(T) = \mathcal{P}_5$

② $T: M_{2\times 3} \longrightarrow \mathcal{P}_2$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \longrightarrow (a+b) + (c+d)x + (e+f)x^2$$

- T is linear
- $\mathcal{N}(T) = ?$ $\qquad (a+b) + (c+d)x + (e+f)x^2 = 0$

gives 3 equations in 6 variables:

$$\begin{cases} a+b = 0 \\ c+d = 0 \\ e+f = 0 \end{cases} \qquad \begin{array}{cccccc} a & b & c & d & e & f \end{array} \quad \left[\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \qquad \begin{array}{l} a = -b \\ c = -d \\ e = -f \end{array}$$

a, c, e dep ; b, d, f indep

so $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} -b & b & -d \\ d & -f & f \end{bmatrix} = b\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$$\mathcal{N}(T) = \text{Sp}\left(\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}\right) \rightsquigarrow \text{nullity}(T) = 3$$

l.i.

- $\mathcal{R}(T) = ?$ Note that $a_0 + a_1 x + a_2 x^2 = T\left(\begin{bmatrix} a_0 & 0 & a_1 \\ 0 & a_2 & 0 \end{bmatrix}\right)$

so $\mathcal{R}(T) = \mathcal{P}_2$. $\rightsquigarrow \text{rank}(T) = 3$

Note: rank(T) + nullity$(T) = 6 = \dim \mathrm{Mat}_{2\times 3}$.

This will be the rank-nullity theorem for abstract vector spaces & linear transf $T: V \to W$ where $\dim V$ is finite.

§3 Key example: Taking coordinates relative to a basis:

Fix V a vector space of dimension p & basis $B = \{\vec{v}_1, \dots, \vec{v}_p\}$

Let
$$T: V \longrightarrow \mathbb{R}^p$$
$$\vec{v} \longmapsto [\vec{v}]_B = \text{coords of } \vec{v} \text{ relative to } B$$

(If $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_p\vec{v}_p$, then $[\vec{v}]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}$

Prop: T is a linear transformation

Proof (1)
$$\vec{v} = a_1\vec{v}_1 + \dots + a_p\vec{v}_p$$
$$+ \quad \vec{w} = b_1\vec{v}_1 + \dots + b_p\vec{v}_p$$
$$\vec{v} + \vec{w} = (a_1+b_1)\vec{v}_1 + \dots + (a_p+b_p)\vec{v}_p$$

so $[\vec{v}+\vec{w}]_B = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_p+b_p \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{bmatrix} = [\vec{v}]_B + [\vec{w}]_B$

(2) $c\vec{v} = (ca_1)\vec{v}_1 + \dots + (ca_p)\vec{v}_p$

so $[c\vec{v}]_B = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_p \end{bmatrix} = c\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} = c[\vec{v}]_B$