Lecture XXVIII: §5.7 Linear Transformations
§5.8 Operations with linear transformations

Recall $T: V \longrightarrow W$ linear transf $\rightsquigarrow$ $\mathcal{N}(T) = \{\vec{v} \text{ in } V : T(\vec{v}) = \vec{0}_W\}$ subspace of V (Nullspace of T)

$\rightsquigarrow$ $\mathcal{R}(T) = \{\vec{w} \text{ in } W : \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \text{ in } V\}$ subspace of W (Range of T)

§1 One-To-One / Onto:

Def T is one-to-one (or injective) if $T(\vec{v}_1) = T(\vec{v}_2)$ implies $\vec{v}_1 = \vec{v}_2$

Def: T is onto (or surjective) if every element of W comes from some element in V (ie $\mathcal{R}(T) = W$. Equivalently $\operatorname{rank}(T) = \dim W$)

Proposition T is one-to-one if and only if $\mathcal{N}(T) = \{\vec{0}_V\}$ (nullity$(T)=0$)

Reason: • Assume T is one-to-one. If $\vec{v}$ in $\mathcal{N}(T)$, then $T(\vec{v}) = \vec{0}_W = T(\vec{0}_V)$

This forces $\vec{v} = \vec{0}_V$. We conclude $\mathcal{N}(T) = \{\vec{0}_V\}$.

• Assume $\mathcal{N}(T) = \{\vec{0}_V\}$ & pick $\vec{v}_1, \vec{v}_2$ with $T(\vec{v}_1) = T(\vec{v}_2)$. Then

$$\vec{0}_W = T(\vec{v}_1) - T(\vec{v}_2) = T(\vec{v}_1) + (-1)T(\vec{v}_2) = T(\vec{v}_1) + T(-\vec{v}_2)$$
$$= T(\vec{v}_1 - \vec{v}_2) \quad \text{so} \quad \vec{v}_1 - \vec{v}_2 \text{ in } \mathcal{N}(T) = \{\vec{0}_V\}.$$

That means $\vec{v}_1 - \vec{v}_2 = \vec{0}$ so $\vec{v}_1 = \vec{v}_2$.

Key example: (Taking coords) Fix V vector space with basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$

Let
$$T: V \longrightarrow \mathbb{R}^p$$
$$\vec{v} \longmapsto [\vec{v}]_{\mathcal{B}} = \text{coords of } \vec{v} \text{ relative to } \mathcal{B}$$

(If $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_p\vec{v}_p$, then $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}$

Prop: T is a linear transformation, 1-to-1 and onto.
(last time)

• To show T is 1-to-1: it's enough to check $\mathcal{N}(T) = \{\vec{0}_V\}$

By construction $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ means $\vec{v} = 0\cdot\vec{v}_1 + 0\cdot\vec{v}_2 + \cdots + 0\cdot\vec{v}_p$
$$= \vec{0}_V + \vec{0}_V + \cdots + \vec{0}_V = \vec{0}_V$$

so $\mathcal{N}(T) = \{\vec{0}_V\}$

- <u>T show T onto</u>: $\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} = T(\vec v)$ where $\vec v = a_1\vec v_1 + \cdots + a_p \vec v_p$

  So any vector in $\mathbb{R}^p$ is in $\mathcal{R}(T)$.

§2. Invertible linear transformations

Fix $T : V \longrightarrow W$ a linear transformation

<u>Definition</u>: T is <u>invertible</u> (or an isomorphism) if T is both 1-to-1 and onto.

<u>Reason</u>: T is 1-to-1 and onto if, and only if, every $\vec v$ in W comes from <u>exactly one</u> $\vec v$ in V

In other words, for every $\vec w$ in W there is a unique $\vec v$ in V with $T(\vec v) = \vec w$

So, we can define a "return to sender" map $\boxed{S : W \longrightarrow V}$ where

$$S(\vec w) = \vec v \quad \text{if } \vec v \text{ is the unique vector in } V \text{ with } T(\vec v) = \vec w$$

It is clear that $S(T(\vec v)) = \vec v$ for every $\vec v$ in V

$T(S(\vec w)) = \vec w$ —— $\vec w$ in W

<u>Proposition</u>: S is a linear transformation

<u>Proof</u>. (1) If $\vec w_1, \vec w_2$ are in W, and $S(\vec w_1) = \vec v_1$, $S(\vec w_2) = \vec v_2$ (meaning $T(\vec v_1) = \vec w_1$ and $T(\vec v_2) = \vec w_2$) then

$$T(\vec v_1 + \vec v_2) = T(\vec v_1) + T(\vec v_2) = \vec w_1 + \vec w_2$$

This means that $S(\vec w_1 + \vec w_2) = \vec v_1 + \vec v_2 = S(\vec w_1) + S(\vec w_2)$. ✓

(2) $S(c\vec w) = c\,S(\vec w)$ is checked similarly:

$S(\vec w) = \vec v$ means $T(\vec v) = \vec w$ so $T(c\vec v) = c\vec w$,

meaning $S(c\vec w) = c\vec v = c\,S(\vec w)$ ✓

Example: ① $T: \mathbb{R} \longrightarrow \mathbb{R}$ linear
$x \longmapsto 2x$

$S: \mathbb{R} \longrightarrow \mathbb{R}$
$y \longmapsto x$ with $T(x) = 2x = y$ $\Rightarrow$ $x = \frac{y}{2}$

Conclusion: $S: \mathbb{R} \longrightarrow \mathbb{R}$ is the inverse to T (it's linear)
$y \longmapsto \frac{y}{2}$

② $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ for a fixed $n\times n$ matrix A
$\vec{x} \longmapsto A\vec{x}$

Q: What can we say about A if T is invertible?

- 1-to-1: $\mathcal{N}(A) = \mathcal{N}(T) = \{0_V\}$, so A is non-singular
- onto: $\mathcal{R}(A) = \mathcal{R}(T) = \mathbb{R}^n$ $\Rightarrow$ $\operatorname{rk} A = n$

Both conditions are equivalent to A being invertible

Conclusion T invertible means A is invertible

The inverse map $S: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is given by
$\vec{y} \longmapsto \vec{x}$ with $A\vec{x} = \vec{y}$

multiplication by the $n\times n$ matrix A^{-1} (so it's a linear map!)

Indeed, $A\vec{x} = \vec{y}$ gives

$$A^{-1}(A\vec{x}) = A^{-1}\vec{y}$$
$$(A^{-1}A)\vec{x} = A^{-1}\vec{y}$$
$$\vec{x} = I_n\vec{x} = A^{-1}\vec{y}$$

§3. Identity Transformation:

Fix a vector space V.

Definition: The identity transformation, denoted by $I_V: V \longrightarrow V$, is the linear transformation $Id_V(\vec{v}) = \vec{v}$ for every $\vec{v}$ in V.

Our calculations in §4 show:

Theorem: Fix vector spaces V, W & a linear transformation $T: V \longrightarrow W$. The following assertions are equivalent to each other:

(1) T is both 1-to-1 and onto

(2) For every $\vec{w}$ in W there is a unique $\vec{v}$ in V with $T(\vec{v}) = \vec{w}$

(3) There is a linear transformation $S: W \longrightarrow V$ with

$S(T(\vec{v})) = \vec{v}$ for each $\vec{v}$ in V $\quad [\, S \circ T = Id_V \,]$

& $T(S(\vec{w})) = \vec{w}$ —— $\vec{w}$ in W $\quad [\, T \circ S = Id_W \,]$

"composition"

§4. Composition of Linear Transformations:

Assume we have 3 vector spaces and 2 linear transformations

$$T_1 : U \longrightarrow V \quad ; \quad T_2 : V \longrightarrow W$$

We can "compose" these just as with ordinary functions of 1 variable

$T_2 \circ T_1 : U \longrightarrow W$ (read from right to left: apply T_1 first and then T_2)

$T_2 \circ T_1(\vec{u}) = T_2(\underbrace{T_1(\vec{u})}_{\text{in } V})$ for any $\vec{u}$ in U

Example:

$T_1 : \mathcal{P}_4 \longrightarrow \mathcal{P}_3$, $f \longmapsto \frac{df}{dx}$ linear

$T_2 : \mathcal{P}_3 \longrightarrow \text{Mat}_{2\times 3}$, $T_2(a + bx + cx^2 + dx^3) = \begin{bmatrix} a & 0 & b \\ c & d & 0 \end{bmatrix}$ linear

$$T_2 \circ T_1(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = T_2\left(\frac{d}{dx}(a_0 + a_1 x + a_2 x^2 + a_3 x^3)\right)$$

$$= T_2(\underbrace{a_1}_{a} + \underbrace{2a_2}_{b} x + \underbrace{3a_3}_{c} x^2) \underset{d=0}{} = \begin{bmatrix} a_1 & 0 & 2a_2 \\ 3a_3 & 0 & 0 \end{bmatrix}$$

§5. Addition & Scalar Multiplication

Just as with matrices, we can ① add two linear transformations between the same vector spaces, and ② scale a linear transf.

① $F : V \longrightarrow W$ linear
$G : V \longrightarrow W$ ——— $\rightsquigarrow$ $F+G : V \longrightarrow W$ linear

$$(F+G)(\vec{v}) = F(\vec{v}) + G(\vec{v})$$
for all $\vec{v}$ in V

② $F : V \longrightarrow W$ linear
c scalar $\rightsquigarrow$ $cF : V \longrightarrow W$ linear

$$(cF)(\vec{v}) = c \cdot F(\vec{v})$$ for all $\vec{v}$ in V

in W

Examples: $F : \mathcal{P}_3 \longrightarrow \mathcal{P}_3$ $\quad F(p) = \dfrac{dp}{dx}$

$G : \mathcal{P}_3 \longrightarrow \mathcal{P}_3$ $\quad G(p) = p(-x)$

$$F(a+bx+cx^2+dx^3) = b+2cx+3dx^2$$
$$G(a+bx+cx^2+dx^3) = a-bx+cx^2-dx^3$$

- So $(F+G)(a+bx+cx^2+dx^3) = b+2cx+3dx^2+a-bx+cx^2-dx^3$
$= (a+b) + (2c-b)x + (c+3d)x^2 - dx^3$

is linear $\quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longmapsto \begin{bmatrix} a+b \\ 2c-b \\ c+3d \\ -d \end{bmatrix}$ linear (at the level of standard coords in $\mathcal{P}_3$)

- $(3F)(a+bx+cx^2+dx^3) = 3b+6cx+9dx^2$ $\quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longrightarrow \begin{bmatrix} 3b \\ 6c \\ 9d \\ 0 \end{bmatrix}$