Lecture XXIX: § 5.9 Matrix representations of linear transf

Recall: $T: V \longrightarrow W$ linear transformation is

- <u>1-to-1</u> if $\mathcal{N}(T) = \{\vec{0}_V\}$
- <u>onto</u> if $\mathcal{R}(T) = W$
- <u>invertible</u> if both 1-to-1 & onto. The inverse map $S: W \longrightarrow V$ is defined as $S(\vec{y}) = \vec{x}$ if $T(\vec{x}) = \vec{y}$.

Key fact: S is also a linear transformation &

(1) $V \xrightarrow{T} W \xrightarrow{S} V$ & (2) $W \xrightarrow{S} V \xrightarrow{T} W$

$S \circ T = Id_V$ $\qquad$ $T \circ S = id_W$

$(S(T(\vec{v})) = \vec{v}$ for all $\vec{v}$ in $V)$ $\qquad$ $(T(S(\vec{w})) = \vec{w}$ for all $\vec{w}$ in $W)$

- Define 3 operations
  - $+$: $F: V \longrightarrow W$, $G: V \longrightarrow W$ $\rightsquigarrow$ $F+G: V \longrightarrow W$, $\vec{v} \longmapsto F(\vec{v}) + G(\vec{v})$
  - scalar multiplication: $\alpha F: V \longrightarrow W$ $\quad (\alpha F)(\vec{v}) = \alpha F(\vec{v})$
  - composition: $U \xrightarrow{F} V \xrightarrow{G} W$ $\quad G \circ F(\vec{u}) = G(\underbrace{F(\vec{u})}_{\text{in } V})$

<u>TODAY</u>: Build matrices representing linear transformation & see how these matrices behave under 3 operations

<u>IDEA</u>: $\boxed{\begin{array}{c}\text{Linear Transf}\\ T: V \longrightarrow W\end{array}} \xrightarrow[\begin{cases} B \text{ for } V \\ C \text{ for } W\end{cases}]{\text{choice of basis}} \boxed{\begin{array}{c} m \times n \text{ matrix } [T]_{CB} \\ m = \dim W,\ n = \dim V\end{array}}$

§ 1. Matrix Representation of a Linear Transformation

We will proceed as we did for $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear

Fix V, W vector spaces with $\dim V = n$, $\dim W = m$

Let $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for V

$C = \{\vec{w}_1, \ldots, \vec{w}_m\}$ ——————— W

Given a linear transformation $T: V \longrightarrow W$, we get an $m \times n$ matrix A as follows.

$$\begin{cases} T(\vec{v}_1) = a_{11}\vec{w}_1 + a_{21}\vec{w}_2 + \cdots + a_{m1}\vec{w}_m \\ T(\vec{v}_2) = a_{12}\vec{w}_1 + a_{22}\vec{w}_2 + \cdots + a_{m2}\vec{w}_m \\ \vdots \\ T(\vec{v}_n) = a_{1n}\vec{w}_1 + a_{2n}\vec{w}_2 + \cdots + a_{mn}\vec{w}_m \end{cases} \rightsquigarrow A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Name: A = matrix representation of T relative to the bases B of V & C of W

Notation: $A = [T]_{CB}$

coordinates of $T(\vec{v}_1)$ relative to the basis C

Observe $A = \left[\ \underset{\text{in } \mathbb{R}^m}{[T(\vec{v}_1)]_C} \ \cdots \ \underset{\text{in } \mathbb{R}^m}{[T(\vec{v}_n)]_C}\ \right]$ $m \times n$ matrix

Columns of A are the coordinates of $T(\vec{v}_i)$ relative to C, where $\vec{v}_i$ is the i^{th} basis element in B

Example $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear; B, C standard bases in $\mathbb{R}^n$ & $\mathbb{R}^m$

then $A = [T(\vec{e}_1) \cdots T(\vec{e}_n)]$

Key: $\underset{m\times 1}{[T(\vec{v})]_C} = \underset{m\times n}{[T]_{CB}}\ \underset{n\times 1}{[\vec{v}]_B}$; Can read of $\mathcal{N}(T)$ & $\mathcal{R}(T)$ from the matrix

Examples: ① $F: \mathcal{P}_3 \longrightarrow \mathcal{P}_2$ $F(p) = \frac{dp}{dx}$

$F(a + bx + cx^2 + dx^3) = b + 2cx + 3dx^2$

Take $B = \{1, x, x^2, x^3\}$ $\dim \mathcal{P}_3 = 4$

$C = \{1, x, x^2\}$ $\dim \mathcal{P}_2 = 3$

$\rightsquigarrow [T]_{BC}$ 3×4 matrix

$F(1) = 0$ $F(x) = 1$ $F(x^2) = 2x$ $F(x^3) = 3x^2$

$[F(1)]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $[F(x)]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $[F(x^2)]_C = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ $[F(x^3)]_C = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

Conclude: $[F]_{CB} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

$$F(a+bx+cx^2+dx^3) = b+2cx+3dx^2 \rightsquigarrow [\quad]_C = \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}$$

$$\begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

② $G : \mathcal{P}_2 \longrightarrow \mathcal{P}_2 \qquad G(a+bx+cx^2) = a - bx + cx^2$

Take $B = C = \{1, x, x^2\}$

$G(1) = 1 \qquad G(x) = -x \qquad G(x^2) = x^2$

$[G(1)]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad [G(x)]_C = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \qquad [G(x^2)]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\rightsquigarrow [G]_{CB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \& \quad \begin{bmatrix} a \\ -b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

③ $F : \underbrace{\mathrm{Mat}_{2\times 2}}_{\dim = 2\cdot 2 = 4} \longrightarrow \underbrace{\mathcal{P}_2}_{\dim = 2+1 = 3} \qquad F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = d + (b-c)x + ax^2$

- Basis for $\mathrm{Mat}_{2\times 2}$: $B = \{E_{11}, E_{12}, E_{21}, E_{22}\} \qquad E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, etc.
- Basis for $\mathcal{P}_2$: $C = \{1, x, x^2\}$

To compute $[F]_{BC}$ = 3×4 matrix we have to write $F(E_{11}), \dots, F(E_{22})$ in terms of C.

$F(E_{11}) = F\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \underset{a=1,\, b=c=d=0}{=} 0 + (0-0)x + 1x^2 = x^2 \rightsquigarrow [F(E_{11})]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$F(E_{12}) = F\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \underset{b=1,\, a=c=d=0}{=} 0 + (1-0)x + 0\cdot x^2 = x \rightsquigarrow [F(E_{12})]_C = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$F(E_{21}) = F\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \underset{c=1\ a=b=d=0}{=} 0 + (0-1)x + 0\cdot x^2 = -x \rightsquigarrow [F(E_{21})]_C = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$F(E_{22}) = F\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \underset{d=1,\, a=b=c=0}{=} 1 + (0-0)x + 0\cdot x^2 = 1 \rightsquigarrow [F(E_{22})]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Conclude: $$[F]_{CB} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Observe:

$$\begin{tikzcd}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} & \mathrm{Mat}_{2\times 2} \arrow[rr, "F"] & & \mathcal{P}_2 \arrow[d, "H"', "inv"] & a_0 + a_1 x + a_2 x^2 \arrow[d, leftrightarrow] \\
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \arrow[u, leftrightarrow] & \mathbb{R}^4 \arrow[u, "G", "inv"'] \arrow[rr, "linear", "H\circ F\circ G"'] & & \mathbb{R}^3 & \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}
\end{tikzcd}$$

G & H depend on our choice of bases B & C.

$$H\circ F\circ G\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = H\circ F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = H\left(d + (b-c)x + ax^2\right) = \begin{bmatrix} d \\ b-c \\ a \end{bmatrix}$$

Note $$[F]_{BC}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} d \\ b-c \\ a \end{bmatrix}$$

So the bottom map $\mathbb{R}^4 \longrightarrow \mathbb{R}^3$ is multiplication by $[F]_{BC}$.

Ex.: $$F\left(\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\right) = 4 + (1-(-1))x + 2x^2 = 4 + 2x + 2x^2$$

$$\left[F\left(\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\right)\right]_C = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \underset{[F]_{CB}}{\underset{\parallel}{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}} \underset{\left[\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\right]_B}{\underset{\parallel}{\begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \end{bmatrix}}}$$

§2. Properties of matrix representations

① Fix $T: \underset{\dim n}{V} \longrightarrow \underset{\dim m}{W}$ linear transf & basis B_V for V & B_W for W.

Then, for every $\vec{v}$ in V, we have:

$$\underset{m\times 1}{[T(\vec{v})]_{B_W}} = \underset{m\times n}{[T]_{B_W B_V}} \; \underset{n\times 1}{[\vec{v}]_{B_V}}$$

In pictures

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "{[\ ]_{B_V}}"'] & W \arrow[d, "{[\ ]_{B_W}}"] \\
\mathbb{R}^n \arrow[r, "{\cdot [T]_{B_W B_V}}"] & \mathbb{R}^m
\end{tikzcd}$$

Proof Write $B_V = \{\vec{v}_1, \dots, \vec{v}_n\}$ & $B_W = \{\vec{w}_1, \dots, \vec{w}_m\}$, then

$$[T]_{B_W B_V} = \Big[\underbrace{[T(\vec{v}_1)]_{B_W}}_{1^{st}\text{ col}} \; \cdots \; \underbrace{[T(\vec{v}_n)]_{B_W}}_{n^{th}\text{ col}} \Big]$$

Write $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n$. Then, using the fact that $[\;]_{B_W}$ is a linear transf, we get

$$\begin{aligned}
[T(\vec{v})]_{B_W} &= [a_1 T(\vec{v}_1) + \dots + a_n T(\vec{v}_n)]_{B_W} \\
&= a_1 \underbrace{[T(\vec{v}_1)]_{B_W}}_{\substack{1^{st}\text{ col} \\ [T]_{B_W B_V}}} + a_2 \underbrace{[T(\vec{v}_2)]_{B_W}}_{\substack{2^{nd}\text{ col} \\ [T]_{B_W B_V}}} + \dots + a_n \underbrace{[T(\vec{v}_n)]_{B_W}}_{\substack{n^{th}\text{ col} \\ [T]_{B_W B_V}}} \\
&= [T]_{B_W B_V} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = [T]_{B_W B_V} [\vec{v}]_B \quad \checkmark
\end{aligned}$$

II Addition & Scalar Mult

$T_1 : V \longrightarrow W$, $T_2 : V \longrightarrow W$, c real number

$$[T_1 + T_2]_{B_W B_V} = [T_1]_{B_W B_V} + [T_2]_{B_W B_V}$$

$$[cT_1]_{B_W B_V} = c[T_1]_{B_W B_V}$$

III Composition of lin transf $\longleftrightarrow$ Matrix multiplication

Pick linear transformations $F : U \longrightarrow V$, $G : V \longrightarrow W$, basis B_U for U, B_V for V, B_W for W

Assume $\dim U = n$, $\dim V = k$, $\dim W = m$

Then $G \circ F : U \longrightarrow W$ satisfies

$$\underset{m \times n}{[G \overset{\text{composition}}{\circ} F]_{B_W B_U}} = \underset{m \times k}{[G]_{B_W B_V}} \overset{\text{matrix multiplication}}{\cdot} \underset{k \times n}{[F]_{B_V B_U}}$$

Reason: The (LHS) satisfies $\left([G]_{B_W B_V} [F]_{B_V B_U}\right)[\vec{u}]_{B_U} = [(G \circ F)(\vec{u})]_{B_W}$

because $\left([G]_{B_W B_V}[F]_{B_V B_U}\right)[\vec{u}]_{B_U} = [G]_{B_W B_V}\left([F]_{B_V B_U}[\vec{u}]_{B_U}\right) =$

$= [G]_{B_W B_V}[F(\vec{u})]_{B_V} = [G(F(\vec{u}))]_{B_W} = [(G\circ F)(\vec{u})]_{B_W}$

IV $T: V \longrightarrow W$ linear $\quad B_V$ basis for $V \quad \dim V = n$

$\qquad\qquad\qquad\qquad\qquad\quad B_W$ ——— $W \quad \dim W = n$

Then:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "{[\ ]_{B_V}}"'] & W \arrow[d, "{[\ ]_{B_W}}"] \\ \mathbb{R}^n \arrow[r, "\tilde{T}", "A = {[T]_{B_W B_V}}"'] & \mathbb{R}^m \end{tikzcd}$$

$\tilde{T}$ = multiplication by A ($m \times n$ matrix)

We can compute $\mathcal{N}(T)$ & $\mathcal{R}(T)$ by working with A

Prop: (1) $\vec{v}$ in $\mathcal{N}(T)$ if and only if $[\vec{v}]_{B_V}$ is in $\mathcal{N}(A)$

(2) $\vec{w}$ in $\mathcal{R}(T)$ ——— $[\vec{w}]_{B_W}$ is in $\mathcal{R}(A)$

In particular:
- nullity(T) = $\dim \mathcal{N}(T) = \dim \mathcal{N}(A) =$ nullity(A)
- rank(T) = $\dim \mathcal{R}(T) = \dim \mathcal{R}(A) =$ rank(A)

Consequence 1: Rank-Nullity Theorem for T

For Matrices : rank (A) + nullity(A) = # Cols(A)

For $T: V \to W$: rank(T) + nullity (T) = $\dim(V)$
linear transf

Consequence 2: $T: V \longrightarrow W$ linear transformation, $\dim V = n$, $\dim W = m$ with bases B for V & C for W respectively. Then, T is invertible if, and only if, $[T]_{CB}$ is invertible (In particular, $m = n$). Furthermore, we have $[T^{-1}]_{BC} = ([T]_{CB})^{-1}$