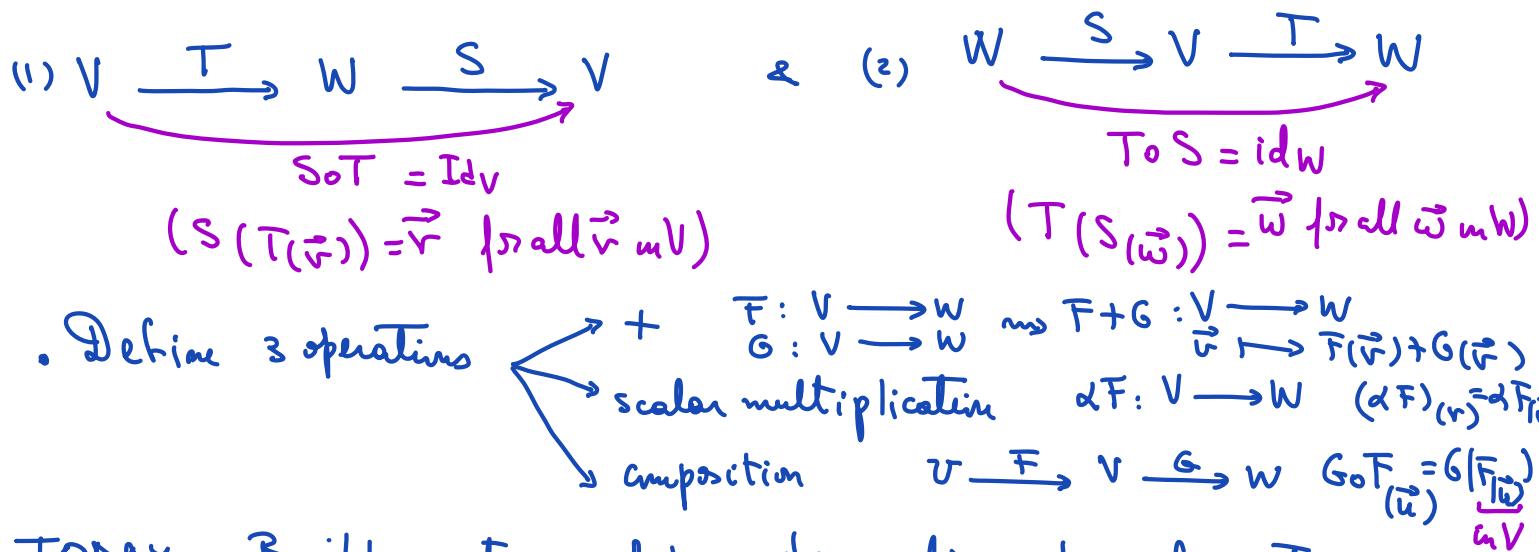


Lecture XXIX: § 5.9 Matrix representations of linear transf

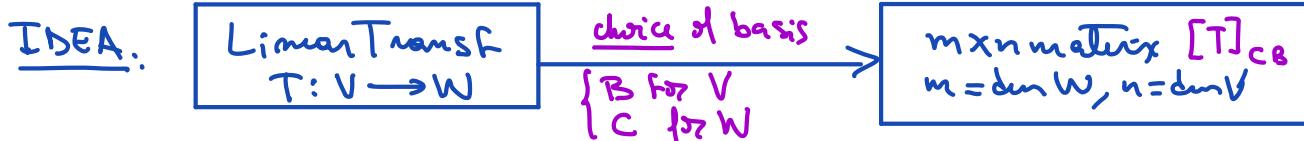
Recall: $T: V \rightarrow W$ linear transformation is

- 1-to-1 if $N(T) = \{0_V\}$
- onto if $R(T) = W$
- invertible if both 1-to-1 & onto. The inverse map $S: W \rightarrow V$ is defined as $S(\vec{y}) = \vec{x}$ if $T(\vec{x}) = \vec{y}$.

Key fact: S is also a linear transformation &



TODAY: Build matrices representing linear transformation & see how these matrices behave under 3 operations



§ 1. Matrix Representation of a Linear Transformation

We will proceed as we did for $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear

Fix V, W vector spaces with $\dim V = n$, $\dim W = m$

Let $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for V

$C = \{\vec{w}_1, \dots, \vec{w}_m\}$ _____ W

Given a linear transformation $T: V \rightarrow W$, we set an $m \times n$ matrix A as follows.

$$\left\{ \begin{array}{l} T(\vec{v}_1) = a_{11} \vec{w}_1 + a_{21} \vec{w}_2 + \dots + a_{m1} \vec{w}_m \\ T(\vec{v}_2) = a_{12} \vec{w}_1 + a_{22} \vec{w}_2 + \dots + a_{m2} \vec{w}_m \\ \vdots \\ T(\vec{v}_n) = a_{1n} \vec{w}_1 + a_{2n} \vec{w}_2 + \dots + a_{mn} \vec{w}_m \end{array} \right. \Rightarrow A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

Name: A = matrix representation of T relative to the bases B of V & C of W

Notation: $A = [T]_{CB}$ wordmatrices of $T(\vec{v}_i)$ relative to the basis C

Observe $A = \left[\begin{array}{c} [T(\vec{v}_1)]_C \\ \vdots \\ [T(\vec{v}_n)]_C \end{array} \right]$ $m \times n$ matrix
 in \mathbb{R}^m in \mathbb{R}^m

Columns of A are the coordinates of $T(\vec{v}_i)$ relative to C , where \vec{v}_i is the i^{th} basis element in B

Example $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear; B, C standard bases in \mathbb{R}^n & \mathbb{R}^m

then $A = [T(\vec{e}_1) \dots T(\vec{e}_n)]$

Key: $\left[T(\vec{v}) \right]_C = \begin{matrix} [T]_{CB} \\ m \times 1 \end{matrix} \quad \left[\vec{v} \right]_B \quad n \times 1$; can read off $N(T)$ & $R(T)$ from the matrix

Examples: ① $F: \mathcal{P}_3 \rightarrow \mathcal{P}_2$ $F(p) = \frac{dp}{dx}$

$$F(a+bx+cx^2+dx^3) = b+2cx+3dx^2$$

$$\text{Take } B = \{1, x, x^2, x^3\} \quad \dim \mathcal{P}_3 = 4$$

$$C = \{1, x, x^2\} \quad \dim \mathcal{P}_2 = 3$$

$\Rightarrow [T]_{BC}$ 3×4 matrix

$$F(1) = 0$$

$$F(x) = 1$$

$$F(x^2) = 2x$$

$$F(x^3) = 3x^2$$

$$[F(1)]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[F(x)]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[F(x^2)]_C = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$[F(x^3)]_C = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Conclude: $[F]_{CB} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

$F(a+bx+cx^2+dx^3) = b+2cx+3dx^2 \rightsquigarrow []_C = \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}$

$\begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

(2) $G: \mathbb{P}_2 \longrightarrow \mathbb{P}_2 \quad G(a+bx+cx^2) = a-bx+cx^2$

Take $B=C = \{1, x, x^2\}$

$G(1) = 1 \quad G(x) = -x \quad G(x^2) = x^2$
 $[G(1)]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [G(x)]_C = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad [G(x^2)]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\rightsquigarrow [G]_{CB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \& \quad \begin{bmatrix} a \\ -b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

dim = 2 · 2 = 4 dim = 2 + 1 = 3

(3) $F: \text{Mat}_{2 \times 2} \longrightarrow \mathbb{P}_2 \quad F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = d + (b-c)x + ax^2$

. Basis for $\text{Mat}_{2 \times 2}$: $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, etc.

. Basis for \mathbb{P}_2 : $C = \{1, x, x^2\}$

To compute $[F]_{BC} = 3 \times 4$ matrix we have to write $F(E_{11}), \dots, F(E_{22})$ in terms of C.

$F(E_{11}) = F\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0 + (0-0)x + 1 \cdot x^2 = x^2 \rightsquigarrow [F(E_{11})]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$a=1, b=c=d=0$

$F(E_{12}) = F\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 0 + (1-0)x + 0 \cdot x^2 = x \rightsquigarrow [F(E_{12})]_C = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

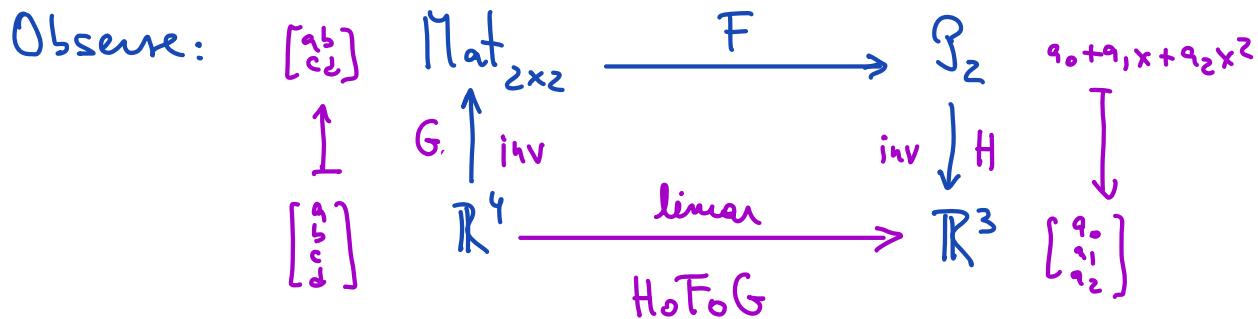
$b=1, a=c=d=0$

$F(E_{21}) = F\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = 0 + (0-1)x + 0 \cdot x^2 = -x \rightsquigarrow [F(E_{21})]_C = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$c=1, a=b=d=0$

$F(E_{22}) = F\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1 + (0-0)x + 0 \cdot x^2 = 1 \rightsquigarrow [F(E_{22})]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\text{Conclude: } [F]_{CB} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



$$H \circ F \circ G \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = H \circ F \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = H \left(d + (b-c)x + ax^2 \right) = \begin{bmatrix} d \\ b-c \\ a \end{bmatrix}$$

Note $[F]_{BC} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} d \\ b-c \\ a \end{bmatrix}$

So the bottom map $\mathbb{R}^4 \longrightarrow \mathbb{R}^3$ is multiplication by $[F]_{BC}$.

Ex. $F \left(\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \right) = 4 + (1 - (-1))x + 2x^2 = 4 + 2x + 2x^2$

$$[F \left(\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \right)]_C = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$[F]_{CB}$ " $[\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}]_B$

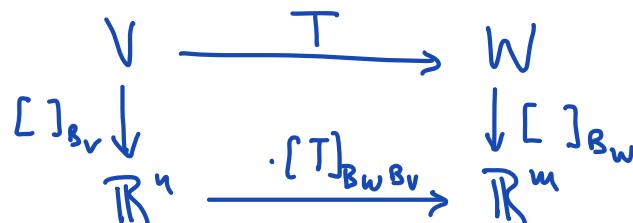
§2. Properties of matrix representations

① Fix $T: V \xrightarrow{\text{dim } n} W \xrightarrow{\text{dim } m}$ linear transf & basis B_V for V & B_W for W

Then, for every $\vec{v} \in V$, we have:

$$[T(\vec{v})]_{B_W}^{m \times 1} = [T]_{B_W B_V}^{m \times n} [\vec{v}]_{B_V}^{n \times 1}$$

In pictures



Proof Write $B_V = \{\vec{v}_1, \dots, \vec{v}_n\}$ & $B_W = \{\vec{w}_1, \dots, \vec{w}_m\}$, then

$$[T]_{B_W B_V} = \begin{bmatrix} [T(\vec{v}_1)]_{B_W} & \dots & [T(\vec{v}_n)]_{B_W} \end{bmatrix}_{\substack{1^{\text{st}} \text{ col} \\ n^{\text{th}} \text{ col}}}$$

Write $\vec{v} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_n \vec{v}_n$ Then, using the fact that $[]_{B_W}$ is a linear transf, we get

$$\begin{aligned} [T(\vec{v})]_{B_W} &= [a_1 T(\vec{v}_1) + \dots + a_n T(\vec{v}_n)]_{B_W} \\ &= a_1 [T(\vec{v}_1)]_{B_W} + a_2 [T(\vec{v}_2)]_{B_W} + \dots + a_n [T(\vec{v}_n)]_{B_W} \\ &= [T]_{B_W B_V} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = [T]_{B_W B_V} [\vec{v}]_B \quad \checkmark \end{aligned}$$

(II) Addition & Scalar Mult $T_1: V \rightarrow W, T_2: V \rightarrow W$ crucial number

$$[T_1 + T_2]_{B_W B_V} = [T_1]_{B_W B_V} + [T_2]_{B_W B_V}$$

$$[c T_1]_{B_W B_V} = c [T_1]_{B_W B_V}$$

(III) Composition of lin transf \longleftrightarrow Matrix multiplication

Pick linear transformations $F: U \rightarrow V$, basis B_U for U
 $G: V \rightarrow W$, basis B_V for V
 $G: V \rightarrow W$, basis B_W for W

Assume $\dim U = n, \dim V = k, \dim W = m$

Then $G \circ F: U \rightarrow W$ satisfies

$$[G \circ F]_{B_W B_U} = [G]_{B_W B_V} \cdot [F]_{B_V B_U} \quad \begin{matrix} \text{composition} \\ m \times n \\ m \times k \\ \underline{k \times n} \end{matrix} \quad \begin{matrix} \text{matrix multiplication} \\ \underline{k \times n} \end{matrix}$$

Reason: The (LHS) satisfies $([G]_{B_W B_V} [F]_{B_V B_U}) [\vec{u}]_{B_U} = [G \circ F](\vec{u})_{B_W}$

because $([G]_{B_W B_V} [F]_{B_V B_U}) [\vec{u}]_{B_U} = [G]_{B_W B_V} ([F]_{B_V B_U} [\vec{u}]_{B_U}) =$
 $= [G]_{B_W B_V} [F(\vec{u})]_{B_V} = [G(F(\vec{u}))]_{B_W} = [(G \circ F)(\vec{u})]_{B_W}$

IV $T: V \rightarrow W$ linear B_V basis for V $\dim V = n$
 B_W basis for W $\dim W = m$

Then:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \downarrow []_{B_V} & & \downarrow []_{B_W} \\ \mathbb{R}^n & \xrightarrow{\tilde{T}} & \mathbb{R}^m \\ A = [T]_{B_W B_V} & & \end{array}$$

\tilde{T} = multiplication by A (matrix)

We can compute $N(T)$ & $R(T)$ by working with A

Prop: (1) \vec{v} in $N(T)$ if and only if $[\vec{v}]_{B_V}$ is in $N(A)$
(2) \vec{w} in $R(T)$ \quad $[\vec{w}]_{B_W}$ is in $R(A)$

In particular: . nullity(T) = $\dim N(T) = \dim N(A) =$ nullity(A)
. rank(T) = $\dim R(T) = \dim R(A) =$ rank(A)

Consequence 1: Rank-Nullity Theorem for T

For Matrices: $\text{rank}(A) + \text{nullity}(A) = \#\text{cols}(A)$

For $T: V \rightarrow W$: $\text{rank}(T) + \text{nullity}(T) = \dim(V)$
linear transf

Consequence 2: $T: V \rightarrow W$ linear transformation, $\dim V = n$,
 $\dim W = m$ with bases B for V & C for W respectively. Then,
 T is invertible if, and only if, $[T]_{CB}$ is invertible (In particular,
 $n = m$). Furthermore, we have $[T^{-1}]_{BC} = ([T]_{CB})^{-1}$