Lecture XXX: §6.2 Determinants

Q: What are determinants?

A: For every $n \geq 2$ we define a function $\det: \mathrm{Mat}_{n\times n} \longrightarrow \mathbb{R}$
$$A \longmapsto \det(A)$$

Main properties:

⓪ $\det(A)$ is a polynomial in the entries of A.

① A is singular (ie., non-invertible) if, and only if, $\det A = 0$
(we saw this for 2x2 matrices) [Determinant Tests Invertibility]

② det is multiplicative ($\det(AB) = \det(A)\det B$)

③ det is compatible with elementary row operations (we can track changes)

④ If A is upper triangular, ie $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn}\end{bmatrix}$, then

$\det A = a_{11}a_{22}\cdots a_{nn}$ = product of diagonal entries

↳ 0's below the diagonal

In particular: $\det I_n = \det\left(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1\end{pmatrix}\right) = \underbrace{1\cdots\cdots 1}_{n \text{ times}} = 1$

⑤ $\det(A^T) = \det A$

We'll see a way to compute $\det A$ via "row expansion". ⑤ says we can also compute it by "expanding by columns"

§2. The 2x2 case:

(Base case for a recursive construction)

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$ 2x2 $\rightsquigarrow$ $\boxed{\det(A) = a_{11}a_{22} - a_{12}a_{21}}$

We check the 6 properties in the 2x2 case:

⓪ $\det A$ is a polynomial in the entries of A

① A invertible if and only if $\det(A) \neq 0$. $\left(A^{-1} = \frac{1}{\det A}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11}\end{pmatrix}\right)$

[If $\det A = 0$, then $\begin{bmatrix} a_{11} \\ a_{21}\end{bmatrix}$ $\begin{bmatrix} a_{12} \\ a_{22}\end{bmatrix}$ are proportional, so l.d. & hence A is singular]

② $\det\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix} e & f \\ g & h\end{bmatrix}\right) = \det\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh\end{pmatrix} = (ae+bg)(cf+dh) -$
$(af+bh)(ce+dg) = \cancel{aecf} + aedh + bgcf + \cancel{bgdh} - \cancel{afce} - afdg$
$- bhce - \cancel{bhdg} = (ad-bc)he + gf(bc-ad) = (ad-bc)(eh-gf) = \det\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right)\det\left(\begin{bmatrix} e & f \\ g & h\end{bmatrix}\right)$

③ (Next Time: §6.3)

④ $\det\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) = ad - b\cdot 0 = ad$

⑤ $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T\right) = \det\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = ad - cb = ad - bc = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

§3. Formulas for Larger matrices:

Recursive approach: Define det for $n\times n$ matrices using determinants of submatrices of size $(n-1)\times(n-1)$, starting from $n=3$

These submatrices are obtained by removing row 1 & each of the n columns of A (one at a time). Name: Cofactor matrices

Note that this is what we did to define cross products in $\mathbb{R}^3$.

General Cofactors

For each $(i\ j)$ — i: row index, j: column index — we define an $(n-1)\times(n-1)$ matrix

M_{ij} = delete row i & col j from A

Eg: $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 4 \\ -1 & 3 & 7 \end{bmatrix} \rightsquigarrow M_{11} = \begin{bmatrix} 0 & 4 \\ 3 & 7 \end{bmatrix}\quad M_{21} = \begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix}\quad M_{12} = \begin{bmatrix} 2 & 4 \\ -1 & 7 \end{bmatrix}$ ...

Def: $A_{ij} = (-1)^{i+j}\det(M_{ij})$ These numbers are called cofactors

Definition: $\det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$ for $n\geq 2$

(expansion of det along 1st row, alternating signs)

$\det([a]) = a \qquad (n=1)$

Remark: This definition is recursive in nature (Base case is $n=1$)

Check 2×2 definition matches the old one:

$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a(-1)^{1+1}\det(d) + b(-1)^{1+2}\det(c) = ad - bc$ ✓

Example 1 $\quad A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 4 \\ -1 & 3 & 7 \end{bmatrix} \quad 3\times 3$

$M_{11} = \begin{bmatrix} 0 & 4 \\ 3 & 7 \end{bmatrix} \rightsquigarrow \det M_{11} = 0\cdot 7 - 4\cdot 3 = -12 \qquad A_{11} = (-1)^{1+1} \det M_{11} = -12$

$M_{12} = \begin{bmatrix} 2 & 4 \\ -1 & 7 \end{bmatrix} \rightsquigarrow \det M_{12} = 2\cdot 7 + 1\cdot 4 = 18 \quad \& \quad A_{12} = (-1)^{1+2} \det M_{12} = -18$

$M_{13} = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} \rightsquigarrow \det M_{13} = 2\cdot 3 - 0\cdot(-1) = 6 \quad \& \quad A_{13} = (-1)^{1+3} \det M_{13} = 6$

So $\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 1\cdot(-12) + 2(-18) + 1\cdot 6 = -12-36+6$
$= -42$

Example 2 $\quad A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & 2 & 3 & 1 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix}$

don't need to compute this

$$\det A = 1 \det \begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & 2 & 3 & 1 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix} - 2 \det \begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & 2 & 3 & 1 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix} + 0 \det \begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & 2 & 3 & 1 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix}$$
$$- 2 \det \begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & 2 & 3 & 1 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & -2 & 1 \end{bmatrix} = 2 \det \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} - 3 \det \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} + 1 \det \begin{bmatrix} 2 & -1 \\ -3 & -2 \end{bmatrix}$$
$$= 2(-1) \quad -3\cdot 2 \quad +1(-4-3) = -2-6-7 = -15$$

$$\det \begin{bmatrix} -1 & 3 & 1 \\ -3 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix} = -1 \det \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} - 3 \det \begin{bmatrix} -3 & 0 \\ 2 & 1 \end{bmatrix} + 1 \det \begin{bmatrix} -3 & -1 \\ 2 & -2 \end{bmatrix}$$
$$= -1\cdot(-1) \quad -3(-3) \quad +1(6+2) = 1+9+8 = 18$$

$$\det \begin{bmatrix} -1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -3 & -2 \end{bmatrix} = -1 \det \begin{bmatrix} 2 & -1 \\ -3 & -2 \end{bmatrix} - 2 \det \begin{bmatrix} -3 & -1 \\ 2 & -2 \end{bmatrix} + 3 \det \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}$$
$$= -1(-4-3) \quad -2(6+2) + 3(9-4) = 7-16+15 = 6$$

So $\det(A) = (-15) - 2\cdot 18 - 2\cdot 6 = -15-36-12 = -63$

Observation: Since the definition of determinants is recursive, any claim about determinants is proven using an induction argument, meaning

- check it for 2×2 case (Base Case)
- assume it for $(n-1)\times(n-1)$ case & check it for $(n\times n)$ case using the previous cases (Inductive Step)

Proposition 1: If 1^{st} Column of A is $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ then $\det(A)=0$.

Proof: • $\det\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = 0$ ✓ so the claim is true for 2×2 matrices

• If $A = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ has size $n\times n$ then

$$\det A = 0 \det\begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ \vdots & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix} - a_{12}\det\begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix} +$$

$$+ a_{13}\det\begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \cdots + (-1)^{1+n}\det\begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$(n-1)\times(n-1)$ matrices with 1^{st} col $= \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

so by $(n-1)\times(n-1)$ case these determinants are $=0$

Conclude $\det A = 0 + 0 = 0$

Property 2: $\det\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} = a_{11}a_{22}\cdots a_{nn}$ (product of diagonal entries)

Proof Use Proposition + induction

• 2×2 case $\det\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = ad - b\cdot 0 = ad$ ✓

- Assume $(n-1)\times(n-1)$ case & use definition

$$\det A = a_{11}\det\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn}\end{bmatrix} - a_{12}\det\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn}\end{bmatrix} + \cdots$$

(upper Triangular)

$$+ (-1)^{1+n}\det\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn}\end{bmatrix}$$

1st col $= \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$$= a_{11}\det\begin{bmatrix} a_{22} & \cdots & a_{2n} \\ 0 & \ddots & a_{nn}\end{bmatrix} \underbrace{- a_{12}\cdot 0 + \cdots + (-1)^{1+n}\,0}_{=0}$$

$(n-1)\times(n-1)$ case

$$= a_{11}\,a_{22}\cdots a_{nn} \quad ✓$$