Lecture XXXI: §6.3 Elementary Row Operations & determinants

Recall: $\det : \mathrm{Mat}_{n\times n} \longrightarrow \mathbb{R}$

$A \longmapsto \det(A)$ a number

Recursive definition:

$n=1$ $\det([a]) = a$

$n\geq 2$ $\det\left(\begin{bmatrix} \boxed{a_{11} \cdots a_{1n}} \\ \vdots \quad \vdots \\ a_{n1} \cdots a_{nn}\end{bmatrix}\right) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$

(signs alternate!)

$A_{ij} := (-1)^{i+j} \det(A \text{ without row } i \text{ \& col } j)$

$(n-1)\times(n-1)$ matrix

Ex $\underline{n=2}$ $\det\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right) = ad - bc$

$\underline{n=3}$ $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9\end{bmatrix}$ $(\det(A) = 0)$ A is singular ($\mathrm{Col}_1 + \mathrm{Col}_3 = 2\,\mathrm{Col}_2$)

$$\det(A) = 1\det\left(\begin{bmatrix} 5 & 6 \\ 8 & 9\end{bmatrix}\right) - 2\det\left(\begin{bmatrix} 4 & 6 \\ 7 & 9\end{bmatrix}\right) + 3\det\left(\begin{bmatrix} 4 & 5 \\ 7 & 8\end{bmatrix}\right)$$
$$= 1(45-48) - 2(36-42) + 3(32-35)$$
$$= 1(-3) - 2(-6) + 3(-3) = -3 + 12 - 9 = 0$$

Properties ✓ ⓪ $\det(A)$ is a polynomial in the entries of A.

① A is singular (i.e., non-invertible) if, and only if, $\det A = 0$
[Determinant Tests Invertibility]

② $\det$ is multiplicative ($\det(AB) = \det(A)\det(B)$)

③ $\det$ is compatible with elementary row operations (we can track changes) ← $\boxed{\text{TODAY}}$

✓ ④ If A is upper triangular, ie $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{21} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn}\end{bmatrix}$, then

0's below the diagonal

$\det A = a_{11}a_{22}\cdots a_{nn}$ = product of diagonal entries

④ bis If A is lower triangular $A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & & \vdots \\ & & \ddots & 0 \\ a_{n1} & \cdots & & a_{nn}\end{bmatrix}$, then

$\det A = a_{11}a_{12}\cdots a_{nn}$

In particular: $\det I_n = \det\left(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1\end{pmatrix}\right) = \underbrace{1 \cdot \cdots \cdot 1}_{n \text{ times}} = 1$

5 $\det(A^T) = \det A$

Remark: Recursive formulas $\rightsquigarrow$ Inductive proof (Base case: $n=2$)

5 is KEY to show 3, 2 will be used in 1

§1. Elementary row Operations

INPUT: $n \times n$ matrix A $\xrightarrow{\text{ROW OPERATIONS}}$ OUTPUT: $n\times n$ matrix B

Q: How are $\det(B)$ & $\det(A)$ related?

A:

Operation	$\det(B)$	Net Effect
(I) SWAP: Exchange 2 Rows $R_i \leftrightarrow R_j$	$-\det(A)$	sign change
(II) SCALE by $\alpha \neq 0$ $R_i \to \alpha R_i$	$\alpha \det(A)$	multiply by scalar α
(III) COMBINE $R_i \to R_i + aR_j$ $(i\neq j)$	$\det(A)$	no change

Why? Use $\det(A^T) = \det(A)$ & use column operations instead with the cofactor (recursive) formula

(I) SWAP $A \xrightarrow[R_i \leftrightarrow R_j \ (i\neq j)]{} B$ [det swaps sign.]

(1) $n=2$: $\det\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right) = ad - bc$ vs $\det\left(\begin{bmatrix} c & d \\ a & b\end{bmatrix}\right) = cb - da = -\det\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right)$

(2) $n=3$ $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5\end{bmatrix}$ $\det A = 1 \det\left(\begin{bmatrix} 1 & 2 \\ 4 & 5\end{bmatrix}\right) - 0 + 2\det\left(\begin{bmatrix} 1 & 1 \\ 3 & 4\end{bmatrix}\right) = (5-8) + 2(4-3)$
$= -3 + 2 = -1$

$\updownarrow -$ (recursion) $\updownarrow -$

$R_2 \leftrightarrow R_3$: $B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 1 & 1 & 2\end{bmatrix}$ $\det B = 1\det\left(\begin{bmatrix} 4 & 5 \\ 1 & 2\end{bmatrix}\right) - 0 + 2\det\left(\begin{bmatrix} 3 & 4 \\ 1 & 1\end{bmatrix}\right) = (8-5) + 2(3-4)$
$= 1$

$R_1 \leftrightarrow R_2$: $\quad B = \begin{bmatrix} 1&1&2\\1&0&2\\3&4&5\end{bmatrix}$ $\quad \det B = \det\left(\begin{bmatrix}0&2\\4&5\end{bmatrix}\right) - \det\left(\begin{bmatrix}1&2\\3&5\end{bmatrix}\right) + 2\det\left(\begin{bmatrix}1&0\\3&4\end{bmatrix}\right)$

$= -8 \;-(5-6) + 2\cdot 4 = -8+1+8 = 1$

Reason behind the rule: 2 ways to check

1. Check rule applied by swapping columns
2. Use $\det(A^T) = \det(A)$

or

3. Show it directly: need to argue separately if Row 1 is swapped or not.

Consequence $\quad \det\left(\begin{smallmatrix}\text{matrix with}\\ \text{2 repeated rows}\end{smallmatrix}\right) = 0 \quad$ (swap 2 repeated rows to get $\det(A) = -\det(A)$, so $\det(A) = 0$)

(II) SCALE $\quad A \xrightarrow[R_i \to \alpha R_i]{} B \quad$ for $\alpha \neq 0 \quad$ [scale by α]

(1) $n=2$ $\quad \det\left(\begin{bmatrix}\alpha a & \alpha b\\ c & d\end{bmatrix}\right) = (\alpha a)d - (\alpha b)c = \alpha(ad-bc) = \alpha\det\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right)$

$\det\left(\begin{bmatrix}a&b\\ \alpha c&\alpha d\end{bmatrix}\right) = a(\alpha d) - b(\alpha c) = \alpha(ad-bc) = \alpha\det\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right)$

(2) $n=3$ $\quad A = \begin{bmatrix}1&0&2\\1&1&2\\3&4&5\end{bmatrix} \qquad \det(A) = -1$

$R_1 \to \alpha R_1 \quad B = \begin{bmatrix}\alpha&0&2\alpha\\1&1&2\\3&4&5\end{bmatrix} = \alpha\det\left(\begin{bmatrix}1&2\\4&5\end{bmatrix}\right) - 0 + 2\alpha\det\left(\begin{bmatrix}1&1\\3&4\end{bmatrix}\right) = \alpha\big((5-8)+2(4-3)\big)$
$= \alpha(-3+2) = -\alpha$

$R_2 \to \alpha R_2 \quad B = \begin{bmatrix}1&0&2\\ \alpha&\alpha&2\alpha\\3&4&5\end{bmatrix} = 1\det\left(\begin{bmatrix}\alpha&2\alpha\\4&5\end{bmatrix}\right) - 0 + 2\det\left(\begin{bmatrix}\alpha&\alpha\\3&4\end{bmatrix}\right) = -\alpha$

$\downarrow\alpha. \qquad \downarrow\alpha.$

Reasons behind the rule: $\quad \det A = a_{11}A_{11} + \cdots + a_{1n}A_{1n}$

1. If we scale Row 1 by α, we scale $a_{11}, \ldots, a_{1n}$ by α in the cofactor formula for $\det(B)$
2. If we scale another rule, we scale $A_{11}, \ldots, A_{1n}$ by in the cofactor formula for $\det(B)$

(III) COMBINE $\quad A \xrightarrow[R_i \to R_i + \alpha R_j]{} B \quad i \neq j \quad$ any number $\alpha \quad$ [no effect]

$n=2$ $\quad \det\left(\begin{bmatrix}a & b\\ c+\alpha a & d+\alpha b\end{bmatrix}\right) = a(d+\alpha b) - b(c+\alpha a) = ad-bc+$

$$+\alpha(ab-ba) = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + \alpha \underbrace{\det\left(\begin{bmatrix} a & b \\ a & b \end{bmatrix}\right)}_{=0} = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

<u>$n=3$</u> $\quad A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \qquad \det(A) = -1$

$R_2 \longrightarrow R_2 - R_1 \qquad B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 4 & 5 \end{bmatrix} \qquad \det(B) = 1 \det\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix} - 0 + 2 \det\begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix}$

$= 5 - 0 + 2(-3) = -1$

$R_1 \longrightarrow R_1 + R_2 \qquad B = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \qquad \det(B) = 2\det\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} - 1 \det\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} + 4 \det\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$

$= 2(5-8) - 1(5-6) + 4(4-3) =$

$= -6 + 1 + 4 = -1$

<u>Reasons behind the rule</u>: Distributive laws on $\mathbb{R}$

$$\det(B) = \det(A) + \det\left(\begin{array}{c} i \\ j \end{array}\begin{bmatrix} \alpha R_j \\ R_j \end{bmatrix}\right)$$

$$\underset{\text{Scale rule}}{=} \det(A) + \alpha \det\left(\begin{array}{c} i \\ j \end{array}\begin{bmatrix} R_j \\ R_j \end{bmatrix}\right) = \det(A) + \alpha\cdot 0 = \det(A)$$

2 repeated rows

§3 Algorithm for computing $\det A$:

<u>IDEA</u>: ① Record effect of row reduction operations on $\det(A)$ as we go from A to $B = EF(A)$

② B is upper triangular, so we can compute $\det(B)$ easily

③ Trace back the operations to compute $\det(A)$ using ① & value of $\det(B)$

<u>Example</u>: $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 4 \\ -1 & 3 & 7 \end{bmatrix}$ Compute $\det(A)$ by putting A into its echelon form.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 4 \\ -1 & 3 & 7 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1}]{\text{(III)}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 2 \\ 0 & 5 & 8 \end{bmatrix} \xrightarrow[R_2 \to R_2/-4]{\text{(II)}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1/2 \\ 0 & 5 & 8 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 5R_2]{\text{(III)}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 21/2 \end{bmatrix} = EF(A)$$

Track changes: $\det A \qquad \det A \qquad -\frac{1}{4}\det A \qquad -\frac{1}{4}\det A$

$EF(A)$ has det $= 1 \cdot 1 \cdot \frac{21}{2}$ so $-\frac{1}{4}\det A = \frac{21}{2}$, ie $\det A = -42$

Summary: Given A of size $n\times n$

- Use Gauss-Jordan elimination to put A into Echelon form

$$A \dashrightarrow B = EF(A)$$

- k swaps
- ℓ scales by non-zero numbers $c_1, \dots, c_\ell$

Then $\underbrace{\det(B)}_{\text{EASY TO COMPUTE}} = (-1)^k c_1 \cdots c_\ell \det(A) \rightsquigarrow \boxed{\det A = \dfrac{(-1)^k \det B}{c_1 \cdots c_\ell}}$

Consequence: A is invertible if and only if $\det A \neq 0$.

Reason: A invertible means $EF(A) = \begin{bmatrix} 1 & * & \cdots & * \\ & 1 & \ddots & \vdots \\ & & \ddots & * \\ 0 & & & 1 \end{bmatrix}$ so $\det(B) = 1$.

Converse: A is not invertible means $EF(A)$ has last row of 0's, so $\det B = 0$.

§3. Det is multiplicative:

Theorem: If A, B have size $n\times n$, then $\det(AB) = \det(A)\det(B)$

We saw this last time for $n=2$

Ex $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ $\qquad \det A = 2\cdot 3 - 2 = 4 \neq 0$

$B = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$ $\qquad \det B = 2 - 3 = -1 \neq 0$

$$AB = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -2+2 & 6-4 \\ -1+3 & 3-6 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & -3 \end{bmatrix} \qquad \det AB = -(4) = -4 = 4(-1)$$

Consequence: If A is invertible $\det(A^{-1}) = \dfrac{1}{\det A}$

Q: Why is the product formula valid for arbitrary n?

Special case: If AB is singular, then A or B are singular (HW)

Consequence: $\det(AB)$ & $\det A \det B$ are both 0

Lemma: If A is singular & B is any $n\times n$ matrix, then AB is also singular

Reason: $\underset{n\times n}{C}$ is singular if and only if C^T is (Recall: $(C^T)^{-1} = (C^{-1})^T$)

- $(AB)^T = B^T A^T$ & if A is singular, so is A^T.
- If A^T is singular, then B^TA^T is singular (pick $\vec{x} \neq \vec{0}$ with $A^T\vec{x} = \vec{0}$. Then $(B^TA^T)\vec{x} = B^T(A^T\vec{x}) = B^T\cdot\vec{0} = \vec{0}$

  so B^TA^T is singular: $\vec{0} \neq \vec{x}$ is in $\mathcal{N}(B^TA^T)$). In particular,

  $AB = (B^TA^T)^T$ is singular.

Proof of the Product Rule:

(1) If A is singular, then AB is also singular so

$\det(AB) = 0 = 0\cdot\det B = \det A \cdot \det B$

(2) If A is invertible, then $A \underset{row}{\sim} I_n$ by a sequence of row operations $\beta \det(A) = \det(I_n) = 1$ so $\beta = \frac{1}{\det A} \neq 0$

BUT(*) The same row operations give $AB \sim B$ so

$\beta \det(AB) = \det B$ Conclude: $\det(AB) = \frac{1}{\beta}\det B = \det A \det B$

(*) Optional: Row operations can be viewed as multiplying on the left by particular matrices. ($R = R_s \cdots\cdots R_2R_1$ if s elem row operations)

2x2 case $\quad A = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$ $\quad$ (I) $\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix} a & b \\ c & d\end{bmatrix} = \begin{bmatrix} c & d \\ a & d\end{bmatrix}$ $(R_1 \leftrightarrow R_2)$

(II) $\begin{bmatrix} \alpha & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} a & b \\ c & d\end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ c & d\end{bmatrix}$ $(R_1 \to \alpha R_1)$; $\begin{bmatrix} 1 & 0 \\ 0 & \alpha\end{bmatrix}\begin{bmatrix} a & b \\ c & d\end{bmatrix} = \begin{bmatrix} a & b \\ \alpha c & \alpha d\end{bmatrix}$ $(R_2 \to \alpha R_2)$

(III) $\begin{bmatrix} 1 & \alpha \\ 0 & 1\end{bmatrix}\begin{bmatrix} a & b \\ c & d\end{bmatrix} = \begin{bmatrix} a+\alpha c & b+\alpha d \\ c & d\end{bmatrix}$ $(R_1 \to R_1 + \alpha R_2)$

$\begin{bmatrix} 1 & 0 \\ \alpha & 1\end{bmatrix}\begin{bmatrix} a & b \\ c & d\end{bmatrix} = \begin{bmatrix} a & b \\ c+\alpha a & d+\alpha b\end{bmatrix}$ $(R_2 \to R_2 + \alpha R_1)$

So if $A \underset{row}{\sim} I_n$ by $RA = I_n$ for some special matrix R $(R = A^{-1})$

Then $AB \sim B$ because $RAB = B$