Lecture XXXII: §4.1–4.2 Eigenvalues & Eigenvectors
§4.4 The Characteristic Polynomial

§1. The Eigenvalue Problem (EV): ("eigen" = "self" in German)

Fix a vector space V (eg $V=\mathbb{R}^n$) & a linear transformation $T: V \longrightarrow V$.

Definition: A non-zero vector $\vec{v}$ in V is called an eigenvector of T if $T(\vec{v}) = \lambda \vec{v}$ for some scalar λ (called the eigenvalue of $\vec{v}$)

Ex: $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with $T(\vec{x}) = A\vec{x}$ for some $n\times n$ matrix A
$A = [T(\vec{e}_1) \cdots T(\vec{e}_n)]$

EV Problem Find $\vec{v} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ & λ with $A\vec{v} = \lambda\vec{v}$.

Equivalent formulation: $A\vec{v} - \lambda\vec{v} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$(A - \lambda I_n)\vec{v} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ & $v \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

EV Problem 2: Find λ number with $\mathcal{N}(A - \lambda I_n) \neq \left\{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}\right\}$

Name: $\boxed{E_\lambda} := \mathcal{N}(A - \lambda I_n)$ = eigenspace for λ (= space of all eigenvectors with eigenvalue λ)
$n\times n$

- Equivalently, find λ where $A - \lambda I_n$ is singular (non-invertible)

Advantage: We can use determinants! [$\det(C) = 0$ means C is singular]

Definition: The characteristic polynomial of A is

$$P_A(t) = \det(A - tI_n) = \det \begin{bmatrix} a_{11}-t & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-t & & \vdots \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn}-t \end{bmatrix}$$ [degree n polynomial in t]

Observations (1) $\chi_A(t)$ is a degree n polynomial in t with leading term $(-1)^n$

(2) Eigenvalues = roots of χ_A $\rightsquigarrow$ at most n of them (counted with multiplicity)

Ex: $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \rightsquigarrow P_A(t) = \begin{vmatrix} 1-t & -2 \\ 3 & 4-t \end{vmatrix} = (1-t)(4-t) - 6 = t^2 - 5t + 6$
$= (t-2)(t-3)$

§2 Examples:

① $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ Eigenvalues? $A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2 & 1 are eigenvalues

$$P_A(t) = \det\left(\begin{bmatrix} 2-t & 0 \\ 0 & 1-t \end{bmatrix}\right) = (2-t)(1-t) = (-1)^2 t^2 - 3t + 2$$

upper Δ

roots: 2 & 1 $\rightsquigarrow$ ($\underline{\underline{\text{only}}}$ eigenvalues)

$$E_1 = \mathcal{N}(A - 1 I_2) = \mathcal{N}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = Sp\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$E_2 = \mathcal{N}(A - 2I_2) = \mathcal{N}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\right) = Sp\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

② $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ Eigenvalues = 1 (?) $\chi_A(t) = \det\left(\begin{bmatrix} 1-t & 1 \\ 0 & 1-t \end{bmatrix}\right) = (1-t)^2$

So 1 is the only eigenvalue (double root)

$$E_1 = \mathcal{N}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = Sp\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \qquad \dim E_1 = 1$$

③ $A = \begin{bmatrix} 5 & -1 \\ 8 & -1 \end{bmatrix}$ Eigenvalues?

$$P_A(t) = \det\left(\begin{bmatrix} 5-t & -1 \\ 8 & -1-t \end{bmatrix}\right) = (5-t)(-1-t) + 8 = t^2 - 4t + 3$$

Roots? Use Quadratic formula! $t^2 + bt + c = 0 \rightsquigarrow t = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2}$

$t^2 - 4t + 3 = 0$ has roots $\dfrac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 3}}{2} = \dfrac{4 \pm \sqrt{4}}{2} = \dfrac{4 \pm 2}{2}$ $\nearrow 3$, $\searrow 1$ (2 roots)

$\rightsquigarrow$ 2 eigenvalues: 1 & 3 (simple roots)

$$E_1 = \mathcal{N}\left(\begin{bmatrix} 4 & -1 \\ 8 & -2 \end{bmatrix}\right) = Sp\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) \qquad \left[\begin{array}{cc|c} 4 & -1 & 0 \\ 8 & -2 & 0 \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_1]{} \left[\begin{array}{cc|c} 4 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x_2 = 4x_1$$

$$E_3 = \mathcal{N}\left(\begin{bmatrix} 2 & -1 \\ 8 & -4 \end{bmatrix}\right) = Sp\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \qquad \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 8 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x_2 = 2x_1$$

$\dim E_1 = \dim E_3 = 1$

<u>Obs:</u> $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$ & in this basis

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \vec{x} \longmapsto A\vec{x}$$

has matrix $[T]_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ (diagonal)

$(T(\vec{v}_1) = 3\vec{v}_1 \quad \& \quad T(\vec{v}_2) = \vec{v}_2)$

④ Eigenvalues can be complex numbers (§4.6)

- $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\chi_A(t) = \det\left(\begin{bmatrix} -t & 1 \\ -1 & -t \end{bmatrix}\right) = t^2+1$ has no real roots

Over the complex numbers $t = \pm\sqrt{-1} = i$

- $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ $P_A(t) = \det\begin{bmatrix} 2-t & -1 \\ 1 & 2-t \end{bmatrix} = (2-t)^2+1$
$= t^2-4t+5$

$\leadsto t^2-4t+5=0$ has solutions $t = \dfrac{4 \pm \sqrt{4^2-20}}{2} = \dfrac{4 \pm \sqrt{-4}}{2} = \dfrac{4 \pm 2i}{2}$
$= 2 \pm i$

§2 Why solve the EV problem?

① Diagonalize linear transformations $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ [§4.7]

(Find a basis $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ of eigenvectors, so
$\vec{v}_1 \downarrow \lambda_1$, $\vec{v}_n \downarrow \lambda_n$ (allowing repetitions)

$[T]_{BB} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ diagonal matrix representing T.)

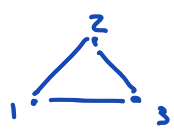
(Last example)

② Solving differential equations $\leadsto$ MATH 2415, 2255

(eg: $T: \mathcal{F} \longrightarrow \mathcal{F}$ where $\mathcal{F}$ = differentiable functions in 1 variable
$f \longmapsto \dfrac{df}{\partial x}$ has eigenvector $f(x) = e^x$ $(f' = e^x)$)

③ Calculating powers of matrices $A, A^2, A^3, \ldots, A^{100}, \ldots$

Ex: Fix a graph G:



A = adjacency matrix 3x3
a_{ij} = # edges between i & j

$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$ $A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ (i,j) entry: paths of length 2 between i & j

$\ldots A^\ell = \underbrace{A \cdots A}_{\ell \text{ times}}$ (i,j) entry = # of paths of length ℓ between i & j

- If A is diagonal, then $A^\ell = \begin{bmatrix} a_{11}^\ell & & 0 \\ & a_{22}^\ell & \\ 0 & & \ddots a_{nn}^\ell \end{bmatrix}$ is easy to do.
- If A is diagonalizable (meaning $T(\vec{x}) = A\cdot\vec{x}$ is diagonalizable) we'll see that A^ℓ is also easy to compute

§3. Properties of Eigenvalues:

Properties: Fix an $n\times n$ matrix A

① A has at most n eigenvalues (counted with multiplicity)

Why? eigenvalues = roots of $P_A(t)$ & $P_A(t)$ has degree n in t.

② If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k (for $k = 1, 2, 3, \ldots$)

Why? $A\vec{v} = \lambda\vec{v}$ implies $A^2\vec{v} = A(A\vec{v}) = A\lambda\vec{v} = \lambda A\vec{v} = \lambda\lambda\vec{v}$
$= \lambda^2\vec{v}$

Similarly $A^k\vec{v} = A^{k-1}(A\vec{v}) = A^{k-1}(\lambda\vec{v}) = \lambda A^{k-1}\vec{v} = \lambda\lambda^{k-1}\vec{v} = \lambda^k\vec{v}$.

③ If A is invertible & λ is an eigenvalue of A, then $\lambda \neq 0$ & $1/\lambda$ is an eigenvalue of A^{-1}. & $E_\lambda(A) = E_{\lambda^{-1}}(A^{-1})$

Why? If $\lambda = 0$ is an eigenvalue then $A\cdot\vec{v} = 0\vec{v} = \vec{0}$ for some $\vec{v} \neq \vec{0}$ so $N(A) \neq \{\vec{0}\}$ meaning A is singular. This is a contradiction!

Now $A\vec{v} = \lambda\vec{v}$ for $\vec{v} \neq \vec{0}$

Multiply by A^{-1} in the left on both sides:

$\vec{v} = A^{-1}A\vec{v} = A^{-1}\lambda\vec{v} = \lambda A^{-1}\vec{v}$

so $\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$ So $\frac{1}{\lambda}$ is an eigenvalue for A^{-1}.

This process can be reversed so the eigenvectors are the same!

Thus: $E_\lambda(A) = E_{\lambda^{-1}}(A^{-1})$.

4 A & A^T have the same eigenvectors because $P_A(t) = P_{A^T}(t)$

$(P_A(t) = \det(A - tI_n) = \det((A - tI_n)^T) = \det(A^T - tI_n) = P_{A^T}(t))$