Lecture XXXIII: §4.5 Eigenvalues and Eigenspaces

Recall: Last time we defined the characteristic polynomial of an $n\times n$ matrix, the eigenvalues & eigenvectors of the matrix.

A $n\times n$ matrix $\rightsquigarrow$ characteristic polynomial of A
(a_{ij})

$$\boxed{P_A(t)} = \det(A - tI_n) = \det\left(\begin{bmatrix} a_{11}-t & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-t & \cdots & a_{2n} \\ & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-t \end{bmatrix}\right)$$

- degree n polynomial in t
- Coefficient of $t^n = (-1)^n$

- λ is an eigenvalue of A $\longleftrightarrow$ $P_A(\lambda) = 0$
  ($A\vec{v} = \lambda\vec{v}$ for some $\vec{v} \neq \vec{0}$) (λ is a root of P_A)
- $E_\lambda(v) = \{\vec{v} \text{ in } \mathbb{R}^n : A\vec{v} = \lambda\vec{v}\} = \mathcal{N}(A - \lambda I_n)$ eigenspace for λ

⚠ A degree n polynomial has at most n roots (but they can be complex or real!). Roots come with multiplicities.

$$P_A(t) = (-1)^n (t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_n)$$

Eigenvalues of A: $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ (real or complex)
§4.5 §4.6

- Algebraic multiplicity of λ = number of times λ appears in the list $\lambda_1, \lambda_2, \ldots, \lambda_n$.
- Geometric multiplicity of λ = $\dim E_\lambda(A)$

Proposition:

$$\boxed{\text{Geometric multiplicity of } \lambda \leq \text{Algebraic multiplicity of } \lambda}$$

Examples (1) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad P_A(t) = \det\left(\begin{bmatrix} 1-t & 1 \\ 0 & 1-t \end{bmatrix}\right) = (1-t)^2 = (t-1)^2$

1 eigenvalue: $\lambda = 1$ $\rightsquigarrow$ algebraic multiplicity $= 2$

$E_1(A) = \mathcal{N}(A - I_2) = \mathcal{N}\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right) = Sp\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ $\begin{cases} y=0\\ 0=0\end{cases}$ solns $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\0\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix}$

$\lambda = 1$ has geometric multiplicity $1 = \dim E_1(A)$. & $1 \leq 2$.

(2) $A = \begin{bmatrix}1&0&-2\\1&3&1\\1&3&1\end{bmatrix}$

$$P_A(t) = \det\left(\begin{bmatrix}1-t&0&-2\\1&3-t&1\\1&3&1-t\end{bmatrix}\right) = (1-t)\det\begin{bmatrix}3-t&1\\3&1-t\end{bmatrix} - 0 + (-2)\det\begin{bmatrix}1&3-t\\1&3\end{bmatrix}$$

$$= (1-t)\big((3-t)(1-t)-3\big) - 2\big(3-(3-t)\big)$$

$$= (1-t)(t^2-4t+3-3) - 2t = t^2-4t-t^3+4t^2-2t$$

$$= -t^3+5t^2-6t = -t(t^2-5t+6) = -t(t-2)(t-3).$$

$\rightsquigarrow$ Eigenvalues: $\lambda = 0, 2$ & 3 all with algebraic multiplicity 1

• $E_0(A) = \mathcal{N}(A - 0I_3) = \mathcal{N}(A) = Sp\left(\begin{bmatrix}2\\-1\\1\end{bmatrix}\right)$

$$\begin{bmatrix}1&0&-2\\1&3&1\\1&3&1\end{bmatrix} \xrightarrow[R_3 \to R_3 - R_2]{} \begin{bmatrix}1&0&-2\\1&3&1\\0&0&0\end{bmatrix} \xrightarrow[R_2 \to R_2 - R_1]{} \begin{bmatrix}1&0&-2\\0&3&3\\0&0&0\end{bmatrix} \xrightarrow[R_3 \to \frac{R_3}{3}]{} \begin{bmatrix}1&0&-2\\0&1&1\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2z\\-z\\z\end{bmatrix} = z\begin{bmatrix}2\\-1\\1\end{bmatrix}$$

• $E_2(A) = \mathcal{N}(A - 2I_3) = \mathcal{N}\left(\begin{bmatrix}-1&0&-2\\1&1&1\\1&3&-1\end{bmatrix}\right) = Sp\left(\begin{bmatrix}-2\\1\\1\end{bmatrix}\right)$

$$\begin{bmatrix}-1&0&-2\\1&1&1\\1&3&-1\end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 + R_1\\ R_3 \to R_3 + R_1}]{} \begin{bmatrix}-1&0&-2\\0&1&-1\\0&3&-3\end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 3R_2\\ R_1 \to -R_1}]{} \begin{bmatrix}1&0&2\\0&1&-1\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-2z\\z\\z\end{bmatrix} = z\begin{bmatrix}-2\\1\\1\end{bmatrix}$$

• $E_3(A) = \mathcal{N}(A - 3I_3) = \mathcal{N}\left(\begin{bmatrix}-2&0&-2\\1&0&1\\1&3&-2\end{bmatrix}\right) = Sp\left(\begin{bmatrix}-1\\1\\1\end{bmatrix}\right)$

All geometric multiplicities are 1

Definition: A is <u>defective</u> if it has an eigenvalue λ with geometric mult λ < algebraic mult of λ
strict

Note: Example (1) was defective & (2) was not.

Q: Why is this relevant?

A: When $\underset{n\times n}{A}$ is non-defective, then $\mathbb{R}^n$ has a basis consisting of eigenvectors for A (We'll see why Next Time!)

Example above: $A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ is non defective

$\mathbb{R}^3$ has basis $B = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ (eigenvectors for A)

eigenvalues : 0 2 3

Def: An $n\times n$ matrix A is called <u>diagonalizable</u> if $\mathbb{R}^n$ has a basis consisting of eigenvectors for A.

Q: Where is the name "diagonalizable" coming from?

Recall: A determines a linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$
$\vec{x} \longmapsto A\vec{x}$

If A is "diagonalizable" we can find a basis B for $\mathbb{R}^n$

$B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ of eigenvectors

Eigenvalues : $\lambda_1, \lambda_2, \ldots, \lambda_n$ (possibly repeated!)

In this situation $T(\vec{v}_1) = \lambda_1 \vec{v}_1$ so $[T(\vec{v}_1)]_B = \begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$,

$T(\vec{v}_2) = \lambda_2 \vec{v}_2$ so $[T(\vec{v}_2)]_B = \begin{bmatrix} 0 \\ \lambda_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, etc.

<u>Conclude</u>: $[T]_{BB} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$ is diagonal.

Note that: $A = [T]_{EE}$ where $E = \{\vec{e}_1, \ldots, \vec{e}_n\}$ standard basis for $\mathbb{R}^n$

<u>Q</u>: How to relate these 2 matrices?

order is important

<u>A</u>: Write $S = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ (put the basis of eigenvectors as columns)

- S is invertible because cols are lin. indep.
- <u>Claim</u>: $S^{-1} A S = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ (Equivalently: $AS = S \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

<u>Why?</u> Compare Columns of both AS & $S \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$$\mathrm{Col}_1(AS) = A\,\mathrm{Col}_1(S) = A\vec{v}_1 = \underbrace{\lambda_1 \vec{v}_1}_{\vec{v}_1 \text{ in } E_{\lambda_1}(A)} = \mathrm{Col}_1\left(S \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\right)$$

$$\mathrm{Col}_2(AS) = A\,\mathrm{Col}_2(S) = A\vec{v}_2 = \lambda_2 \vec{v}_2 = \mathrm{Col}_2\left(S \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\right)$$

Same is true for the rest of the columns.

<u>Conclusion</u>: A is <u>diagonalizable</u> if we can find an invertible matrix S such that $S^{-1}AS = D$ is a diagonal matrix.

In this case: Columns of S = basis of eigenvectors of A

Entries of D = Eigenvalues of A

<u>Example</u>: $A = \begin{bmatrix} 5 & -1 \\ 8 & -1 \end{bmatrix}$ $\quad P_A(t) = \det\left(\begin{bmatrix} 5-t & -1 \\ 8 & -1-t \end{bmatrix}\right) = t^2 - 4t + 3 = (t-3)(t-1)$

$\lambda = 1, 3$ are the only eigenvalues (both have alg mult $=1$)

<u>Last Time</u>: $E_1(A) = \mathcal{N}\left(\begin{bmatrix} 4 & -1 \\ 8 & -2 \end{bmatrix}\right) = Sp\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$

$E_3(A) = \mathcal{N}\left(\begin{bmatrix} 2 & -1 \\ 8 & -4 \end{bmatrix}\right) = Sp\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

Geom mult of 1 & 3 is also 1, so A is not defective

$B = \left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ is a basis of eigenvectors, so A is diagonalizable

$S = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}$ so $S^{-1} = \frac{1}{\det S}\begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix}$

$\rightsquigarrow S^{-1} A S = D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

Advantage of diagonalizable matrices: computing power is easy!

- $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \rightsquigarrow D^2 = \begin{bmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{bmatrix}, D^3 = \begin{bmatrix} \lambda_1^3 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^3 \end{bmatrix}, \ldots, D^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}$

- $S^{-1}AS = D \rightsquigarrow A = SDS^{-1}$

$A^2 = (SDS^{-1})^2 = (SDS^{-1})(SDS^{-1}) = SD\underbrace{S^{-1}S}_{I_n}DS^{-1} = SD^2S^{-1}$

$A^3 = (SDS^{-1})(SDS^{-1})^2 = SDS^{-1}SD^2S^{-1} = SD^3S^{-1}$

<u>In general</u>: $A^k = SD^kS^{-1}$ & $D^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}$ for any $k \geq 0$

<u>Example</u>: $A = \begin{bmatrix} 5 & -1 \\ 8 & -1 \end{bmatrix}$ $\quad D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad S = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}$ $\quad S^{-1} = \frac{1}{2}\begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix}$

$A^6 = SD^6S^{-1} = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3^6 \end{bmatrix}\frac{1}{2}\begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 1457 & -364 \\ 2912 & -727 \end{bmatrix}$

<u>Next Time</u>: We'll see why non-defective matrices are diagonalizable

Properties of eigenvalues (in HW10)

① If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k (for $k = 1, 2, 3, \ldots$)

② If A is invertible & λ is an eigenvalue of A, then $\lambda \neq 0$ & $1/\lambda$ is an eigenvalue of A^{-1}. & $E_\lambda(A) = E_{\lambda^{-1}}(A^{-1})$

③ A & A^T have the same eigenvectors because $P_A(t) = P_{A^T}(t)$

④ If λ is an eigenvalue of A & β is any number, then $(\lambda+\beta)$ is an eigenvalue of $A+\beta I_n$.

Why?

$$
\begin{array}{rl}
 & A\vec{v} = \lambda\vec{v} \qquad \vec{v} \neq \vec{0} \\
+ & \beta\vec{v} = \beta\vec{v} \\
\hline
\end{array}
$$

$(A+\beta I_n)\vec{v} = A\vec{v} + \beta\vec{v} = (\lambda+\beta)\vec{v}$ $\quad \vec{v} \neq \vec{0}$ so $\vec{v}$ is an eigenvector with eigenvalue $\lambda+\beta$.

<u>Example</u> $A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ $\qquad P_A(t) = -\lambda(\lambda-2)(\lambda-3)$

- A singular ($\lambda = 0$ is an eigenvalue)
- $0^2, 2^2, 3^2$ are eigenvalues of A^2, $0, 2^3, 3^3$ are eigenvalues of A^3
- Eigenvalues of $A - 5I_3 = \begin{bmatrix} -4 & 0 & -2 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{bmatrix}$ are $0-5, 2-5, 3-5$
  $= -5, -3, -2$