Lecture XXXV: §4.6 Complex Numbers & complex vector spaces

TODAY'S GOAL: Define complex numbers & its use in the EV Problem

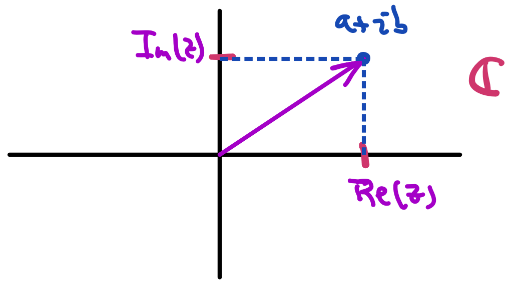
§1. Complex Numbers = $\mathbb{C}$

The set $\mathbb{C}$ has $+$ & multiplication operations (defined below)

① $\mathbb{C}$ is a set enlarging $\mathbb{R}$ that contains the roots of all polynomials in $\mathbb{R}[x]$ (Example: $x^2+1 = (x+i)(x-i)$ $\quad i^2=-1$)

② $\mathbb{C}$ can be identified with $\mathbb{R}^2$ & gives a multiplication operation to $\mathbb{R}^2$

Definition: A complex number z corresponds to a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$, written as $\quad z = a+ib \qquad i =$ placeholder



Names: $a = \mathrm{Re}(z) =$ "real part of z"

$b = \mathrm{Im}(z) =$ "imaginary part of z"

Obs: Two complex numbers agree when the corresponding real & imaginary parts match. $\quad a+ib = c+id \longleftrightarrow a=c \;\&\; b=d$

• $\mathbb{R} \subset \mathbb{C}$: $\quad a = a+i\cdot 0 \quad$ (real line = x-axis in the picture)

• Addition: $(a+ib)+(c+id) = (a+c)+i(b+d)$

(Add Real & Imaginary parts separately)

Example: $(1+i)+(2+i3) = (1+2)+i(1+3) = 3+i4$

• Multiplication: $(a+ib)\cdot(c+id) = (ac-bd)+i(ad+bc)$

In particular: $i^2 = (0+i)(0+i) = (0-1)+i(0) = -1.$

Example: $(1+i)\cdot(2+i3) = (1\cdot 2-1\cdot 3)+i(1\cdot 3+1\cdot 2) = -1+i5$

Q: Why this formula? It's the unique way to make it associative, distrib. commutative, extending multiplication on $\mathbb{R}$ & to have $i^2=-1$

$$(a+ib)\cdot(c+id) \underset{\text{Distribute}}{=} (ac+iad)+ibc+ibid$$

$$\underset{\text{rearrange + regroup}}{=} (ac+i(ad+bc)) + \underbrace{i^2}_{=-1}bd$$

$$\underset{\text{regroup}}{=} (ac-bd)+i(ad+bc)$$

- Modulus: $z = a+ib \rightsquigarrow |z| = \sqrt{a^2+b^2}$ modulus of z, $|z|\geq 0$

It is the magnitude of the vector $\begin{bmatrix} a \\ b \end{bmatrix}$

Key Property: $\boxed{|z\cdot w| = |z|\,|w|} \longleftrightarrow |z\cdot w|^2 = |z|^2|w|^2$

Why? $z = a+ib \rightsquigarrow |z| = \sqrt{a^2+b^2}$

$w = c+id \rightsquigarrow |w| = \sqrt{c^2+d^2}$

$$zw = (a+ib)(c+id) = (ac-bd)+i(cd+bc)$$

$$\rightsquigarrow |zw| = \sqrt{(ac-bd)^2+(ad+bc)^2}$$

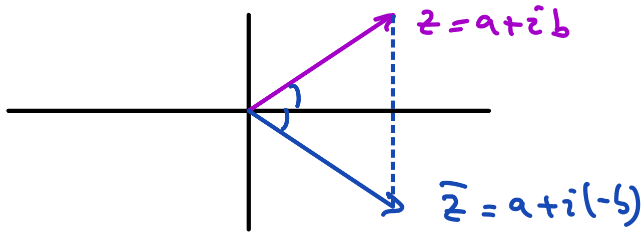
$$|zw| = \sqrt{(a^2c^2+b^2d^2-\cancel{2abcd})+(a^2d^2+b^2c^2+\cancel{2adbc})}$$

$$= \sqrt{a^2(c^2+d^2)+b^2(d^2+c^2)} = \sqrt{(a^2+b^2)(c^2+d^2)} = |z|\,|w|$$

§2. New operation: Complex Conjugation:

Definition: Given $z = a+ib$, its complex conjugate is $\bar{z} = a-ib$
$= a+i(-b)$

Visually:



conjugate = mirror image about x-axis

In particular: $z=\bar{z}$ if and only if z is a real number $(b=0)$

Properties: 1) $\overline{z+w} = \bar{z}+\bar{w}$

Why? $z = a+ib \rightsquigarrow \bar{z} = a+i(-b)$; $\bar{w} = c+i(-d)$

$z+w = (a+c)+i(b+d) \rightsquigarrow \overline{z+w} = a+c + i(-b-d) = (a+i(-b)) + (a+i(-d))$
$= \bar{z} + \bar{w}$

(2) $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

Why? $\overline{z \cdot w} = \overline{(ac-bd)+i(ad+bc)} = (ac-bd) + i(-ad-bc)$

$\bar{z} \cdot \bar{w} = (a+i(-b)) \cdot (c+i(-d)) = (ac-bd) + i(-ad-bc)$ $\parallel$ ✓

(3) $z \cdot \bar{z} = |z|^2$

Why? $z = a+ib$ $\quad z \cdot \bar{z} = (a+ib)(a+i(-b)) = a^2-(-b^2) + i(-ab+ba)$
$\bar{z} = a+i(-b)$ $\quad = a^2+b^2 = |z|^2$.

Consequence: Every $z \neq 0$ in $\mathbb{C}$ has an inverse: $\boxed{z^{-1} = \dfrac{\bar{z}}{|z|^2}}$

($z \neq 0$ means either $\mathrm{Re}(z)$ or $\mathrm{Im}(z) \neq 0$ so $|z| > 0$)

Application: Use this to write ratios of complex numbers as a complex number.

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id} \cdot \frac{c-id}{c-id} = \frac{(ac-bd)+i(-ad+bc)}{c^2+d^2} = \frac{ac-bd}{c^2+d^2} + i\frac{(-ad+bc)}{c^2+d^2}$$

multiply & divide by complex conjugate

EXAMPLES (1) $z = 1+i$ $\quad \bar{z} = 1-i$ $\quad |z|^2 = 1^2+1^2 = 2 > 0$

so $z^{-1} = \dfrac{\bar{z}}{|z|^2} = \dfrac{1-i}{2} = \dfrac{1}{2} - \dfrac{i}{2}$ $\quad$ (Check: $(1+i)(\frac{1}{2}-\frac{i}{2}) = 1$)

(2) $\dfrac{2+i}{1+i} = \dfrac{2+i}{1+i} \; \dfrac{1-i}{1-i} = \dfrac{(2+1)+i(-2+1)}{2} = \dfrac{3}{2} - \dfrac{i}{2}$

(3) $z = 1-i3 \rightsquigarrow \bar{z} = 1+i3$
$w = 2+i4 \rightsquigarrow \bar{w} = 2-i4$ $\Big\} \rightsquigarrow \bar{z}\bar{w} = (2+12) + i(-4+6)$
$= 14 + i2$

$z+w = 3+i \rightsquigarrow \overline{z+w} = 3-i = \bar{z}+\bar{w}$

$z \cdot w = (2+12) + i(4-6) = 14 - i2 \rightsquigarrow \overline{zw} = 14 + i2 = \bar{z} \cdot \bar{w}$

Conclusion: "Difficulty" over operating with $\mathbb{C}$ vs $\mathbb{R}$ is "arithmetic is trickier"

§2. Roots of Polynomials:

Fundamental Theorem of Algebra: Every non-constant polynomial in one variable over $\mathbb{C}$ has all its roots in $\mathbb{C}$ ("$\mathbb{C}$ is algebraically closed")

In particular: f in $\mathbb{C}[x]$ of degree $n>0$ has a factorization

$$f(x) = a(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$$ where $\lambda_1, \ldots, \lambda_n$ are the roots.

Example: Quadratic Polynomials

$P(x) = ax^2+bx+c$ $\quad a, b, c$ in $\mathbb{C}$ $\quad a \neq 0$

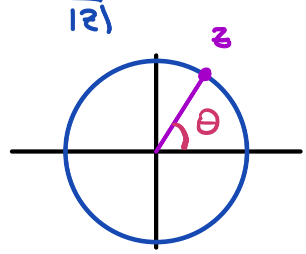
Roots: $= \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ in $\mathbb{C}$ (via quadratic formula)

Q: What does $\sqrt{}$ mean in $\mathbb{C}$?

Ex: $\sqrt{-1} = i$, $\sqrt{-4} = 2\sqrt{-1} = 2i$

In general: $z = |z| \cdot \dfrac{z}{|z|}$ → has modulus 1 $\leadsto \sqrt{z} = \sqrt{|z|} \cdot \sqrt{w}$ where $w = \dfrac{z}{|z|}$ has modulus 1

If w in $\mathbb{C}$ has $|w|=1$, it lies in the unit circle in $\mathbb{R}^2$



$\leadsto w = \cos\theta + i \sin\theta$ for $0 \leq \theta \leq 2\pi$

$$\sqrt{w} = \cos\frac{\theta}{2} + i \sin\frac{\theta}{2}$$

Check: $\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = \underbrace{\left(\cos\frac{\theta}{2}\right)^2 - \left(\sin\frac{\theta}{2}\right)^2}_{=\cos\left(2\cdot\frac{\theta}{2}\right)} + i\underbrace{\left(2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right)}_{\sin\left(2\frac{\theta}{2}\right)} = w$

Q: What about polynomials with real coefficients?

A: 2 Types of roots (1) real roots
(2) complex roots: they come in conjugate pairs!

Why? $f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ with $c_0, \ldots, c_n$ in $\mathbb{R}$

If $z = a + ib$ is a root:

$$0 = c_n (a+ib)^n + c_{n-1} (a+ib)^{n-1} + \cdots + c_1 (a+ib) + c_0$$

Take complex conjugation & use the properties to simplify the expression:

$$0 = \overline{0} = \overline{c_n (a+ib)^n + c_{n-1} (a+ib)^{n-1} + \cdots + c_1 (a+ib) + c_0}$$

$$= \overline{c_n} (a-ib)^n + \overline{c_{n-1}} (a-ib)^{n-1} + \cdots + \overline{c_1} (a-ib) + \overline{c_0}$$

$$= c_n (a-ib)^n + c_{n-1} (a-ib)^{n-1} + \cdots + c_1 (a-ib) + c_0 = f(a-ib)$$

real coefficients

so $a-ib$ is also a root of f.