Lecture XXXVI: §4.6 Complex vector spaces, complex eigenvalues

Last Time: Defined complex numbers $z = a+ib$ $\quad a = \mathrm{Re}(z),\ b = \mathrm{Im}(z)$ in $\mathbb{R}$

- $(a+ib)+(c+id) = (a+c)+i(b+d)$
- $(a+ib)\cdot(c+id) = (ac-bd)+i(ad+bc)$ $\rightsquigarrow$ $i^2=-1$
- Complex conjugation $\overline{a+ib} = a+i(-b) = a-ib$ $\rightsquigarrow$ (1) $\overline{z+w} = \overline{z}+\overline{w}$ (2) $\overline{zw} = \overline{z}\cdot\overline{w}$
- Modulus: $|a+ib| = \sqrt{a^2+b^2}$
- $z\overline{z} = |z|^2$, so if $z\neq 0$, then $z^{-1} = \dfrac{\overline{z}}{|z|}$

Fundamental Thm of Algebra: A degree n polynomial in $\mathbb{C}[x]$ has exactly n roots in $\mathbb{C}$ (counted with multiplicity)

Special case: If f in $\mathbb{R}[x]$, its roots come in 2 types:

(1) real roots $\quad$ (2) non-real roots come in conjugate pairs $(\alpha, \overline{\alpha})$.

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -2 & 1 \end{bmatrix} \rightsquigarrow P_A(t) = \det\left(A - tI_3\right) = \det\left(\begin{bmatrix} 1-t & 0 & 0 \\ 0 & 3-t & 1 \\ 0 & -2 & 1-t \end{bmatrix}\right)$

$$= (1-t)\left((3-t)(1-t)+2\right)$$
$$= (1-t)\left(t^2-4t+5\right)$$

Roots: $\dfrac{-(-4)\pm\sqrt{4^2-4\cdot 5}}{2} = \dfrac{4\pm\sqrt{-4}}{2} = \dfrac{4\pm 2\sqrt{-1}}{2} = 2\pm i$ $\quad 2+i,\ 2-i$ complex conjugate

Eigenspaces for $2+i$ & $2-i$? They will be in $\mathbb{C}^3$.

§1 Vectors in $\mathbb{C}^n$

The ideas & algorithms we developed for $\mathbb{R}^n$ will translate directly to $\mathbb{C}^n$.

Def: $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{C}^n$ means $v_1, v_2, \ldots, v_n$ in $\mathbb{C}$

- Addition: entry-by-entry
- Scalar multiplication: scalars are now in $\mathbb{C}$ & we operate entry-by-entry

Example: $\vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2+i \\ 4-i \end{bmatrix}$ $\rightsquigarrow$ $\vec{v}+w = \begin{bmatrix} 1+(2+i) \\ i+(4-i) \end{bmatrix} = \begin{bmatrix} 3+i \\ 4 \end{bmatrix}$

$$i\vec{v} = \begin{bmatrix} i \\ i^2 \end{bmatrix} = \begin{bmatrix} i \\ -1 \end{bmatrix}$$

⚠ Dot Product on $\mathbb{C}^n$ needs to be modified from $\mathbb{R}^n$ one.

$$\vec{v}\cdot\vec{w} = \overline{v_1}\cdot w_1 + \overline{v_2}\cdot w_2 + \cdots + \overline{v_n}\cdot w_n = \overline{\vec{w}\cdot\vec{v}} \quad \text{(not symmetric!)}$$

Example: $\vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} 2+i \\ 4-i \end{bmatrix}$

$$\vec{v}\cdot\vec{w} = \overline{1}\,(2+i) + \overline{i}\,(4-i) = (2+i) - i(4-i) = 2+i-4i-1 = 1-i3$$

$$\vec{w}\cdot\vec{v} = \overline{2+i}\cdot 1 + \overline{(4-i)}\cdot i = 2-i + (4+i)i = 2-i+4i-1 = 1+i3$$

Q: Why this formula?

A: We want $\vec{v}\cdot\vec{v} = |\vec{v}|^2$ to be a non-negative real number & the formula from $\mathbb{C}^n$ restricted to $\mathbb{R}^n$ should agree with the old one.

$$|\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{\overline{v_1}\cdot v_1 + \overline{v_2}\cdot v_2 + \cdots + \overline{v_n}\cdot v_n} = \sqrt{\underbrace{|v_1|^2 + \cdots + |v_n|^2}_{\geq 0}}$$

- **THEOREM 1** $\mathbb{C}^n$ with this addition & scalar multiplication is a vector space over $\mathbb{C}$. It satisfies the same 10 properties defining $\mathbb{R}^n$ as a vector space but now using scalars in $\mathbb{C}$ (see Lecture 14)

- Same ideas from $\mathbb{R}^n$ allow us to define

① Subspaces of $\mathbb{C}^n$: Sets $\mathbb{V}$ satisfying 3 properties:

(S1) $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ is in $\mathbb{V}$

(S2) If $\vec{u}$, $\vec{v}$ are in $\mathbb{V}$, then so is $\vec{u}+\vec{v}$

(S3) If $\vec{u}$ is in $\mathbb{V}$ & a is in $\mathbb{C}$, then $a\vec{u}$ is in $\mathbb{V}$.

② $\bullet\ Sp_{\mathbb{C}}(\vec{v}_1, \dots, \vec{v}_p)$ = all $\mathbb{C}$-linear combinations of $\vec{v}_1, \dots, \vec{v}_p$

(Prototype of a subspace) $= \{ a_1 \vec{v}_1 + \dots + a_p \vec{v}_p : a_1, \dots, a_p \text{ in } \mathbb{C} \}$ arbitrary

$\bullet\ \{\vec{v}_1, \dots, \vec{v}_p\}$ <u>spans</u> $\mathbb{W}$ if $\mathbb{W} = Sp_{\mathbb{C}}(\vec{v}_1, \dots, \vec{v}_p)$

③ <u>$\mathbb{C}$-linear independence</u>: $\{\vec{v}_1, \dots, \vec{v}_p\}$ are l.i if

$$a_1 \vec{v}_1 + \dots + a_p \vec{v}_p = \vec{0}$$

has only 1 solution in $a_1, \dots, a_p$, namely $a_1 = a_2 = \dots = a_p = 0$

Solve the system with matrix $\left[\begin{array}{ccc|c} \vec{v}_1 & \cdots & \vec{v}_n & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right]$. This augmented matrix has entries in $\mathbb{C}$! Gauss-Jordan works verbatim but now we operate with complex numbers.

④ <u>Bases</u>: $B = \{\vec{v}_1, \dots, \vec{v}_p\}$ is a basis for a subspace $\mathbb{W}$ of $\mathbb{C}^n$ if B spans $\mathbb{W}$ & B is linearly indep $\rightsquigarrow \#B = \dim_{\mathbb{C}} \mathbb{W}$.

§2. Abstract Vector Spaces over $\mathbb{C}$:

- They are defined using the same 8 properties, but now we use scalars in $\mathbb{C}$. There are 2 main examples:

① $\text{Mat}_{m \times n}(\mathbb{C})$ = $m \times n$ matrices with entries in $\mathbb{C}$

Addition: entry-by-entry, Scalar Mult: entry-by-entry

$\mathbb{O}$ = zero matrix of size $m \times n$

<u>Example</u>: $\begin{bmatrix} i & 1 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ i & 4 \end{bmatrix} = \begin{bmatrix} i & 2 \\ -1+i & 7 \end{bmatrix}$

$$i \begin{bmatrix} i & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & i \\ -i & 3i \end{bmatrix}$$

Basis = $\{E_{11}, \dots, E_{mn}\}$ (same as for $\text{Mat}_{m \times n}(\mathbb{R})$)

(2) $\mathcal{P}_n(\mathbb{C}) = \{p(x) = a_0 + a_1 x + \cdots + a_n x^n : a_0, a_1, \ldots, a_n \text{ in } \mathbb{C}\}$

Usual formulas for addition & scalar multiplication (term-by-term)

Example: $(1+ix) + (2-2ix^2) = 3 + ix - 2ix^2$; $i(1+ix) = i - x$

Basis $= \{1, x, x^2, \ldots, x^n\}$ (same as for $\mathcal{P}_n = \mathcal{P}_n(\mathbb{R})$)

§3 Eigenvectors in $\mathbb{C}^n$:

Pick A = $n\times n$ matrix with real entries & λ in $\mathbb{C}$ an eigenvalue (ie $P_A(\lambda) = 0$). Then:

$E_\lambda := \{\vec{v} \text{ in } \mathbb{C}^n : A\vec{v} = \lambda\vec{v}\} = \mathcal{N}(A - \lambda I_n)$ is a subspace of $\mathbb{C}^n$.

Key: If λ in $\mathbb{R}$, the E_λ has a basis with all vectors in $\mathbb{R}^n$, so

$$\dim_{\mathbb{R}}(\underbrace{E_\lambda \cap \mathbb{R}^n}_{\text{old eigenspace}}) = \dim_{\mathbb{C}} E_\lambda$$

(we are allowing more scalars, but the dimension doesn't change!)

Q: How to find $\mathcal{N}(A - \lambda I_n)$?

A: Use Gauss-Jordan to put $A - \lambda I_n$ in REF allowing scalars in $\mathbb{C}$.

Example: $A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ $\quad P_A(t) = \det\begin{bmatrix} 3-t & 1 \\ -2 & 1-t \end{bmatrix} = t^2 - 4t + 5$

Roots: $\dfrac{-(-4) \pm \sqrt{16 - 4\cdot 5}}{2} = \dfrac{4 \pm \sqrt{-4}}{2} = \dfrac{4 \pm 2\sqrt{-1}}{2} = 2 \pm i$

$$E_{2+i} = \mathcal{N}(A - (2+i)I_2) = \mathcal{N}\left(\begin{bmatrix} 3-(2+i) & 1 \\ -2 & 1-(2+i) \end{bmatrix}\right)$$

$$= \mathcal{N}\left(\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix}\right) \qquad |1-i|^2 = 2$$

$$\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix} \xrightarrow[R_2 \to R_2 + (1-i)R_1]{} \begin{bmatrix} 1-i & 1 \\ 0 & 0 \end{bmatrix} \qquad (1-i)x + y = 0 \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ i-1 \end{bmatrix}$$

$$E_{2+i} = Sp\left(\begin{bmatrix} 1 \\ 1-i \end{bmatrix}\right)$$

$$E_{2-i} = \mathcal{N}\left(A-(2-i)I_2\right) = \mathcal{N}\left(\begin{bmatrix} 3-(2-i) & 1 \\ -2 & 1-(2-i) \end{bmatrix}\right)$$

$$= \mathcal{N}\left(\begin{bmatrix} 1+i & 1 \\ -2 & 1+i \end{bmatrix}\right)$$

complex conjugate of the other matrix!

$$\begin{bmatrix} 1+i & 1 \\ -2 & 1+i \end{bmatrix} \xrightarrow[R_2 \to R_2+(1-i)R_1]{} \begin{bmatrix} 1+i & 1 \\ 0 & 0 \end{bmatrix} \quad (1+i)x+y=0 \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

So $E_{2-i} = Sp\left(\begin{bmatrix} 1 \\ 1+i \end{bmatrix}\right)$

Observation: $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in E_{2+i}, then $\overline{\vec{v}} = \begin{bmatrix} \overline{v_1} \\ \overline{v_2} \end{bmatrix}$ in $E_{\overline{2+i}} = E_{2-i}$

This is true whenever A has REAL entries.

Advantage, E_{2-i} comes for free once we compute E_{2+i}.

RULE: If A $n\times n$ matrix has real entries and $\lambda, \overline{\lambda}$ are complex roots of $P_A(t)$ (recall roots of real polynomials come in conjugate pairs!) then

$B = \{\vec{v}_1, \dots, \vec{v}_p\}$ is a basis for E_λ if, and only if,

$\overline{B} = \{\overline{\vec{v}_1}, \dots, \overline{\vec{v}_p}\}$ ——— $E_{\overline{\lambda}}$

Why? If $A\vec{v} = \lambda\vec{v}$, $\vec{v} \neq \vec{0}$, then $\overline{A\vec{v}} = \overline{\lambda\vec{v}}$, so $\overline{\vec{v}}$ is in $E_{\overline{\lambda}}$.

$$A\overline{\vec{v}} = \overline{A}\,\overline{\vec{v}} = \overline{\lambda}\,\overline{\vec{v}}$$

Conclude: Once we compute B, computing $\overline{B}$ is very easy!

Example 2: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad P_A(t) = \det\begin{bmatrix} -t & 1 \\ -1 & -t \end{bmatrix} = t^2+1$

$$= (t-i)(t+i)$$

Eigenvalues: i & $-i$, each with algebraic multiplicity 1.

$$E_i = \mathcal{N}(A - iI_2) = \mathcal{N}\left(\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\right) = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : \begin{array}{l} -ix + y = 0 \\ -x - iy = 0 \end{array}\right\}$$

1^{st} eqn = $i(2^{nd}$ eqn$)$ so solns $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -ix \end{bmatrix} = x\begin{bmatrix} 1 \\ -i \end{bmatrix}$

$$E_i = Sp\left(\begin{bmatrix} 1 \\ -i \end{bmatrix}\right) \rightsquigarrow E_{-i} = Sp\left(\overline{\begin{bmatrix} 1 \\ -i \end{bmatrix}}\right) = Sp\left(\begin{bmatrix} 1 \\ i \end{bmatrix}\right)$$

geom mult for i & $-i$ is 1

Conclude: A is non-defective & eigenvalues are in $\mathbb{C}$.

, A is diagonalizable over $\mathbb{C}$, but not over $\mathbb{R}$!

$$S^{-1}AS = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \text{with } S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \qquad \det S = 2i$$

$$S^{-1} = \frac{1}{2i}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}$$ (same rule to invert as with real entries)

Consequence: If A is real symmetric, then all eigenvalues are real.

Why? Pick λ in $\mathbb{C}$ root of $P_A(t)$ & $\vec{v}$ in $\mathbb{C}^n$, $\vec{v} \neq \vec{0}$ with $A\vec{v} = \lambda\vec{v}$.

- Take $A\vec{v} = \lambda\vec{v}$ & multiply each side by $\overline{\vec{v}}^T$.

$$\overline{\vec{v}}^T A\vec{v} = \overline{\vec{v}}^T \lambda \vec{v} = \lambda\, \overline{\vec{v}}^T \vec{v} = \boxed{\lambda \|\vec{v}\|^2} \quad (1)$$

But $\overline{\vec{v}}^T A\vec{v} = \overline{\vec{v}}^T(\lambda\vec{v}) = [\overline{v_1}, \dots \overline{v_n}]\begin{bmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{bmatrix} = [\lambda v_1, \dots \lambda v_n]\begin{bmatrix} \overline{v_1} \\ \vdots \\ \overline{v_n} \end{bmatrix} =$

$$(\lambda\vec{v})^T\overline{\vec{v}} = (A\vec{v})^T\overline{\vec{v}} = \vec{v}^T A^T \overline{\vec{v}} = \vec{v}^T A\overline{\vec{v}} = \vec{v}^T\overline{\lambda}\,\overline{\vec{v}} = \overline{\lambda}[v_1 \dots v_n]\begin{bmatrix} \overline{v_1} \\ \vdots \\ \overline{v_n} \end{bmatrix}$$

$$= \overline{\lambda}\,\vec{v}^T\overline{\vec{v}} = \boxed{\overline{\lambda}\|\vec{v}\|^2} \quad (2)$$

Compare (1) & (2) to get $\lambda \underbrace{\|\vec{v}\|^2}_{\neq 0} = \overline{\lambda}\underbrace{\|\vec{v}\|^2}_{\neq 0}$, ie $\lambda = \overline{\lambda}$, meaning λ in $\mathbb{R}$.