5.2-6 $S = \{\vec{V} \text{ in } \mathbb{R}^4 : v_1 + v_4 = 0\}$

Since S is a subset of $\mathbb{R}^4$ properties (a1), (a2), (m1), (m2), (m3) and (m4) are satisfied already

So consider (c1) and (c2)

(C1) let $\vec{V}$ and $\vec{u} \in S$, then $v_1 + v_2 = 0$ and $u_1 + u_2 = 0$

now $(v_1 + u_1) + (v_2 + u_2) = (v_1 + v_2) + (u_1 + u_2)$
$= 0$

(C2) let $\vec{V} \in S$ and $\alpha \in \mathbb{R}$, then $v_1 + v_2 = 0$

now $\alpha\vec{V} \Rightarrow (\alpha v_1) + (\alpha v_2) = \alpha(v_1 + v_2)$
$= \alpha(0) = 0$

$\therefore$ S is a vector space

(5.2-10) $\mathbb{P} = \{P(x) \text{ in } P_2 : P(x) = P(-x) \ \forall x\}$

Again because $\mathbb{P}$ is a subset of P_2

properties (a1), (a2), (m1), (m2), (m3)

and (m4) are satisfied already

so consider (C1) and (C2)

(C1)

let $P(x)$ and $q(x) \in \mathbb{P}$, then
$P(x) = P(-x)$ and $q(x) = q(-x)$

now $(p+q)(x) - (p+q)(-x)$
$= p(x) + q(x) - p(-x) - q(-x)$
$= p(x) + q(x) - p(x) - q(x) = 0$
$\therefore (p+q)(x) = (p+q)(-x)$

(C2) let $P(x) \in \mathbb{P}$ and $\alpha \in \mathbb{R}$

$(\alpha P)(x) - (\alpha P)(-x) = \alpha P(x) - \alpha P(-x)$

$= \alpha\big(p(x) - p(-x)\big) = \alpha(0) = 0$

$\therefore \mathbb{P}$ is a vector space

5.2 - 36) $V = \left\{ \vec{X} : \vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ where } x_2 > 0 \right\}$

Notice $\vec{V}$ is a subset of $\mathbb{R}$ which is a vector space, then the properties of addition and scalar multiplication are satisfies

so consider (C1) and (C2)

(C1) let $\vec{U}$ and $\vec{V} \in V$, then

$$\vec{U} + \vec{V} = \begin{bmatrix} u_1 + v_1 \\ u_2 v_2 \end{bmatrix}$$, since $v_2 > 0$ and v_2 $u_2 v_2 > 0$

(C2) let $\vec{U} \in V$ and $c \in \mathbb{R}$, then

$$c\vec{U} = \begin{bmatrix} cu_1 \\ u_2^c \end{bmatrix}$$ since $u_2 > 0$ $u_2^c > 0$

$\therefore$ V is a vector space

(5.3-4) $W = \{A \text{ in } V : a_{11} a_{12} a_{13} = 0\}$

(S1) $A = O$, then $a_{11} a_{12} a_{13} = 0$
since all the elements are zero

(S2) let A and $B \in W$

now $A+B$ so $(a_{11}+b_{11})(a_{12}+b_{12})(a_{13}+b_{13})$

$$= (a_{11}a_{12} + a_{11}b_{12} + b_{11}a_{12} + b_{11}b_{12})(a_{13}+b_{13})$$

$$= \cancelto{0}{a_{11}a_{12}a_{13}} + a_{11}a_{12}b_{13} + a_{11}b_{12}a_{13}$$

$$+ a_{11}b_{12}b_{13} + b_{11}a_{12}a_{13} + b_{11}a_{12}b_{13}$$

$$+ b_{11}b_{12}a_{13} + \cancelto{0}{b_{11}b_{12}b_{13}}$$

the remaining term may or may not be zero so W is not a subspace

5.3-28 A $(n \times n)$ matrix

Let $B = (A + A^T)/2$

Consider $B^T = \left[(A + A^T)/2 \right]^T = \dfrac{(A+A^T)^T}{2}$

$$= \frac{A^T + A}{2} = \frac{A + A^T}{2} = B$$

so is symmetric

Let $C = (A - A^T)/2$

Consider $C^T = \left[\dfrac{A - A^T}{2} \right]^T = \dfrac{(A - A^T)^T}{2}$

$$= \frac{A^T - A}{2} = -\left(\frac{A - A^T}{2} \right) = -C$$

so is skew symmetric

5.4-4 $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$W = \{A : b = a - c,\ d = 2a + c\}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & a-c \\ c & 2a+c \end{bmatrix} = a\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + c\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

consider $\quad c_1\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + c_2\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$c_1 = 0$

$c_1 - c_2 = 0 \quad \Rightarrow \quad c_1 = c_2 = 0$

$c_2 = 0$

$2c_1 + c_2 = 0 \quad \therefore$ L.I.

Since a and c are arbitrary these matrices span the space of (2×2) matrices

$\therefore \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \right\}$ is a basis

5.4-24

$$[A_1]_B = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} \quad [A_2]_B = \begin{bmatrix} -2 \\ 1 \\ 2 \\ -1 \end{bmatrix} \quad [A_3]_B = \begin{bmatrix} -1 \\ -1 \\ 1 \\ -3 \end{bmatrix}$$

$$[A_4]_B = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

Consider $C = \begin{bmatrix} 1 & -2 & -1 & -2 \\ 2 & 1 & -1 & 2 \\ -1 & 2 & 1 & 2 \\ 3 & -1 & -3 & 0 \end{bmatrix}$

take $C^T = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & 1 & 2 & -1 \\ -1 & -1 & 1 & -3 \\ -2 & 2 & 2 & 0 \end{bmatrix} \begin{array}{c} 2 \\ \\ \\ \\ \end{array}$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 5 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 6 & 0 & 6 \end{bmatrix} \begin{array}{c} \\ -1/5 \\ \\ 1/6 \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 5 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \\ 1/5 \\ \\ \\ \end{array} \Rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

then

$\vec{w}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$ $\vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ $\vec{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

so $B_1 = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ $B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ $B_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

5.4-28 $Q = \{P_1(x), P_2(x), P_3(x)\}$

$P_1(x) = -1 + x + 2x^2$
$P_2(x) = x + 3x^2$
$P_3(x) = 1 + 2x + 8x^2$

$c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x) = a_0 + a_1 x + a_2 x^2$

$$\begin{array}{c} \\ 2 \\ \\ \end{array} \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$\begin{array}{c} \\ -3 \\ \\ \end{array} \left[\begin{array}{ccc|c} -1 & 0 & 1 & a_0 \\ 0 & 1 & 3 & a_0 + a_1 \\ 0 & 3 & 10 & 2a_0 + a_2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} -1 & 0 & 1 & a_0 \\ 0 & 1 & 3 & a_0 + a_1 \\ 0 & 0 & 1 & -a_0 - 3a_1 + a_2 \end{array}\right]$$

$$\begin{array}{c} \\ \\ -1,\,-3 \end{array}\left[\begin{array}{ccc|c} -1 & 0 & 1 & a_0 \\ 0 & 1 & 3 & a_0 + a_1 \\ 0 & 0 & 1 & -a_0 - 3a_1 + a_2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} -1 & 0 & 0 & 2a_0 + 3a_1 - a_2 \\ 0 & 1 & 0 & 4a_0 + 10a_1 - 3a_2 \\ 0 & 0 & 1 & -a_0 - 3a_1 + a_2 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2a_0 - 3a_1 + a_2 \\ 0 & 1 & 0 & 4a_0 + 10a_1 - 3a_2 \\ 0 & 0 & 1 & -a_0 - 3a_1 + a_2 \end{array}\right]$$

$$[p(x)]_{Q} = \begin{bmatrix} -2a_0 - 3a_1 + a_2 \\ 4a_0 + 10a_1 - 3a_2 \\ -a_0 - 3a_1 + a_2 \end{bmatrix}$$