5.7-2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$T(A) = a + 2b - c + d$$

Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ $\quad A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$T(A_1 + A_2) = (a_1 + a_2) + 2(b_1 + b_2) - (c_1 + c_2) + (d_1 + d_2)$$

$$= (a_1 + 2b_1 - c_1 + d_1) + (a_2 + 2b_2 - c_2 + d_2)$$

$$= T(A_1) + T(A_2)$$

Let α be any scalar

$$T(\alpha A) = (\alpha a) + 2(\alpha b) - (\alpha c) + (\alpha d)$$

$$= \alpha(a + 2b - c + d) = \alpha T(A)$$

$\therefore$ is a linear transformation

5.7-12

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a + 2d \\ b - c \end{bmatrix}$$

a) Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$T(A_1 + A_2) = \begin{bmatrix} (a_1 + a_2) + 2(d_1 + d_2) \\ (b_1 + b_2) - (c_1 + c_2) \end{bmatrix} = \begin{bmatrix} (a_1 + 2d_1) + (a_2 + 2d_2) \\ (b_1 - c_1) + (b_2 - c_2) \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + 2d_1 \\ b_1 - c_1 \end{bmatrix} + \begin{bmatrix} a_2 + 2d_2 \\ b_2 - c_2 \end{bmatrix} = T(A_1) + T(A_2)$$

Let α be any scalar

$$T(\alpha A) = \begin{bmatrix} (\alpha a) + 2(\alpha d) \\ (\alpha b) - (\alpha c) \end{bmatrix} = \alpha \begin{bmatrix} a + 2d \\ b - c \end{bmatrix} = \alpha T(A)$$

$\therefore$ is a linear transformation

b) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a + 2d \\ b - c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$a + 2d = 0 \qquad a = -2d$

$b - c = 0 \qquad b = c$

$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\left(\begin{bmatrix} -2d & c \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

c) $\begin{bmatrix} -2d & c \\ c & d \end{bmatrix} = c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} -2 & 6 \\ 0 & 1 \end{bmatrix}$

since c and d are arbitrary, then is a spanning set

now consider $c_1\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_2\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

clearly $c_1 = c_2 = 0$ so are L.I.

$\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $\mathcal{N}(T)$

d) nullity $= \dim \mathcal{N}(T) = 2$

$\dim V = 4 = \dim \mathcal{N}(T) + \dim \mathcal{R}(T)$

$\dim \mathcal{R}(T) = 2$

e) since $\dim R(T) = \dim \mathbb{R}$ then $R(T) = \mathbb{R}^2$

f) let $\vec{v} \in \mathbb{R}^2$ s.t. $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

now consider $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+2d \\ b-c \end{bmatrix}$

$a + 2d = x \qquad a = x - 2d$ for any c and d
$b - c = y \qquad b = y + c$

let $c = d = 1$

$$A = \begin{bmatrix} x-2 & y+1 \\ 1 & 1 \end{bmatrix}$$

5.7-14

$$T(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4)$$

$$= (a_0 - a_1 + 2a_2 - a_3 + a_4)$$
$$+ (-a_0 + 3a_1 - 2a_2 + 3a_3 - a_4)x$$
$$+ (2a_0 - 3a_1 + 5a_2 - a_3 + a_4)x^2$$
$$+ (3a_0 - a_1 + 7a_2 + 2a_3 + 2a_4)x^3$$

$$= a_0(1 - x + 2x^2 + 3x^3) + a_1(-1 + 3x - 3x^2 - x^3)$$
$$+ a_2(2 - 2x + 5x^2 + 7x^3) + a_3(-1 + 3x - x^2 + 2x^3)$$
$$+ a_4(1 - x + x^2 + 2x^3)$$

$$C = \begin{bmatrix} 1 & -1 & 2 & -1 & 1 \\ -1 & 3 & -2 & 3 & -1 \\ 2 & -3 & 5 & -1 & 1 \\ 3 & -1 & 7 & 2 & 2 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 1 & -1 & 2 & 3 \\ -1 & 3 & -3 & -1 \\ 2 & -2 & 5 & 7 \\ -1 & 3 & -1 & 2 \\ 1 & -1 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 5/2 & 7/2 \\ 0 & 1 & -1/4 & 5/4 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5.7-20

$T: U \to V$ be a linear transformation with U finite dimensional, $\dim U = k \in \mathbb{Z}^+$

assume $\dim U > \dim V = r \in \mathbb{Z}^+$

then the matrix A that define T is $r \times k$ for $k > r$, so then the solution to

$A\vec{x} = \vec{0}$ is not trivial and $\dim N(T) > 0$

$\therefore$ T is not one-to-one

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 5/2 & -1/4 & 1 & 0 & 0 \\ 7/2 & 5/4 & 3/2 & 1 & 0 \end{bmatrix}$$

$$\left\{ \begin{array}{l} P_1 = 1 - x + \frac{5}{2}x^2 + \frac{7}{2}x^3 \\ P_2 = x - \frac{1}{4}x^2 + \frac{5}{4}x^3 \\ P_3 = x^2 + \frac{3}{2}x^3 \\ P_4 = x^3 \end{array} \right\}$$ is a basis for P_3

$\dim R(T) = 4$ $\quad \dim P_4 = 5$ so $\dim N(T) = 1$

so is not one-to-one since the null space contain other element besides the zeroe element