Lecture I

§1.1: Matrices and Systems of Linear Equations

Object of study: Simplest functions of several variables (2 or more) = LINEAR!
↳ [MATH 2153]

Eg: $x+2y+z=10$ described a plane with normal direction $\langle 1,2,1\rangle$ passing through $(10,0,0)$.

Def: A linear equation in n unknowns is an equation that can be written as

$$\boxed{a_1x_1+a_2x_2+\cdots+a_nx_n=b} \quad (*)$$

- $a_1,\ldots,a_n$ are the coefficients (fixed numbers in $\mathbb{R}$, $\mathbb{C}$ ...)
- b is the constant term (fixed)
- $x_1,x_2,\ldots,x_n$ are the unknowns (for $n=3$, write $x_1=x$, $x_2=y$, $x_3=z$)

Why linear? Each term has degree 1 in each variable $x_1,x_2,\ldots,x_n$

A solution to $(*)$ is a tuple of numbers $(s_1,\ldots,s_n)$ satisfying $(*)$ where $x_1=s_1,\ldots,x_n=s_n$.

Eg above: $(1,5,1)$ is a solution, $(10,0,0)$ is another solution....

Non-example: $x_1+2\sin x_2=0$ → not linear in x_1,x_2, BUT linear in $(x_1,\tilde{x}_2)$ where $\tilde{x}_2=\sin x_2$.

§2. Linear systems:

GOAL: Find simultaneous (or joint) solutions to $\underbrace{\text{several linear equations}}$
call them systems of linear equations

Eg:

2 eqns, 2 unknowns

$$(\mathrm{I})\ \begin{cases}2x+4y=18\\ x-y=0\end{cases}$$

$x=y=3$ is A solution
In fact, it is the only one

2 eqns, 3 unknowns

$$(\mathrm{II})\ \begin{cases}2x+4y+\boxed{z}=18\\ x-y=0\end{cases}$$

- $(3,3,0)$ is a solution
- $(3,3,1)$ ————
- Infinitely many solutions $=(3,3,a)$ for any a in $\mathbb{R}$

2 eqns, 2 unknowns

$$(\mathrm{III})\ \begin{cases}2x+4y=0\\ x+2y=1\end{cases}$$

NO solutions
($2*2^{nd}$ eqn $=1^{st}$ eqn)
- for coefficients
- not for the constant term

2 eqns, 2 unknowns

$$(\mathrm{IV})\ \begin{cases}2x+4y=2 & (1)\\ x+2y=1 & (2)\end{cases}$$

→ Infinitely many solutions, of the form $(1-2a,a)$ for a in $\mathbb{R}$

Why? From (2): $x=1-2y$ & substitute in (1): $2x+4y=2-4y+4y=2$
so we have NO restrictions for y!

Note: (I) represents $\underline{\underline{2}}$ lines in $\mathbb{R}^2$ meeting at a single point, precisely the solution to the $\underline{\underline{2}}$ equations

(II) represents 2 planes in $\mathbb{R}^3$ that are <u>NOT</u> parallel, so they meet along a line ($\Rightarrow$ infinitely many solutions!)

(III) represents 2 parallel lines in $\mathbb{R}^2$ that are different, so they don't intersect ($\Rightarrow$ no solution!)

(IV) $Eq(2) = 2 * Eq(1)$ so both equations represent the same line in $\mathbb{R}^2$ ($\Rightarrow$ infinitely many points are solutions!)

Def: An $\underline{\underline{m}} \times n$ <u>system</u> of <u>linear equations</u> is a set of $\underline{\underline{m}}$ linear equations in n unknowns:

$$
(**)\quad \left\{\begin{array}{lr}
a_{11} \boxed{x_1} + a_{12} \boxed{x_2} + \cdots + a_{1n} \boxed{x_n} = b_1 & (Eq\ 1) \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2 & (Eq\ 2) \\
\vdots \qquad\qquad\qquad\qquad \vdots \qquad \vdots & \vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m & (Eq\ m)
\end{array}\right.
$$

- $m \cdot n$ <u>coefficients</u> $a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, \ldots a_{mn}$ (fixed numbers)
- n <u>constant terms</u> $b_1, \ldots, b_n$

$\searrow \left\{ a_{ij} : \begin{array}{l} i = \text{indicates eqn number} \\ j = \text{" the unknown} \end{array} \right.$

$\searrow b_{\underline{\underline{i}}}$ where i indicates the eqn number.

A <u>solution</u> to $(**)$ is a tuple $(s_1, \ldots, s_n)$ of numbers that simultaneously satisfies each of the m equations

<u>Example</u> (1) Display the system of eqns with coefficients $a_{11} = 1$, $a_{13} = 2$, $a_{21} = -1$, $a_{22} = 1$, $a_{24} = 5$ and constants $b_1 = 0$, $b_2 = -1$.

$$
\begin{array}{l}
\underline{1}\, x_1 + \underline{\ \ }\, x_2 + \underline{2}\, x_3 + \underline{\ \ }\, x_4 = \underline{0} \\
\underline{-1}\, x_1 + \underline{1}\, x_2 + \underline{\ \ }\, x_3 + \underline{5}\, x_4 = \underline{-1}
\end{array}
\rightsquigarrow
\left\{\begin{array}{l}
x_1 + 2x_3 = 0 \\
-x_1 + x_2 + 5x_4 = -1
\end{array}\right.
$$

empty boxes have coefficient $= 0$.

(c) Verify that $(-2, 3, 1, -\frac{6}{5})$ is a solution.

$x_1 = -2$, $x_2 = 3$, $x_3 = 1$, $x_4 = -\frac{6}{5}$ $\rightsquigarrow$ $\begin{cases} -2 + 2\cdot 1 = 0 \checkmark \\ -(-2) + 3 + 5\left(-\frac{6}{5}\right) = -1 \checkmark \end{cases}$

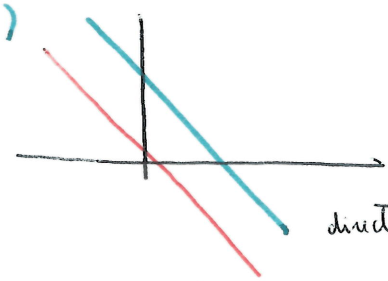
§3 Geometric Interpretation of Solution Sets

(2×2) Systems :
$$\begin{cases} a_{11} x_1 + a_{12} x_2 = b_1 \\ a_{21} x_1 + a_{22} x_2 = b_2 \end{cases}$$
a_{11}, a_{12} not both zero.
a_{21}, a_{22} ————— .

. Each eqn : a line in the plane $\mathbb{R}^2$. ($(a_{11}, a_{12}) \neq (0,0)$, $(a_{21}, a_{22}) \neq (0,0)$)

. Solution : points where 2 lines meet. $\Rightarrow$ 3 possibilities :

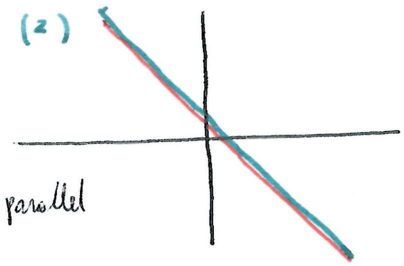
(1)



directions $= (a_{12}, -a_{11})$ are parallel
$(a_{22}, -a_{21})$

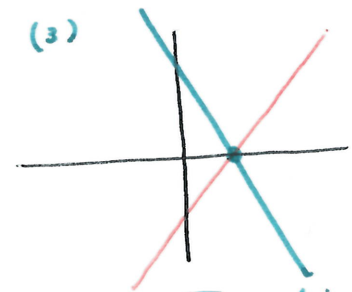
2 parallel lines
(no solution!)

(2)



2 coinciding lines
(infinitely many solns!)

(3)



2 intersecting lines
(unique soln!)

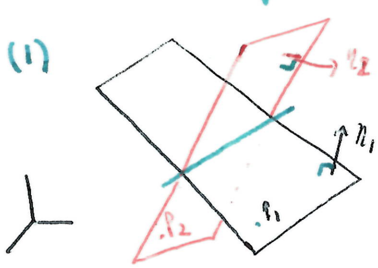
[same number of unknowns than equations = 3 possible number of solutions; none, one or infinitely many]

(2×3) Systems
$$\begin{cases} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2 \end{cases}$$
a_{11}, a_{12}, a_{13} not all 0
a_{21}, a_{22}, a_{23} ————

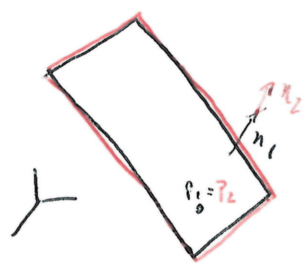
. Each eqn : a plane in the space $\mathbb{R}^3$ (normal vectors $\langle a_{11}, a_{12}, a_{13} \rangle$, $\langle a_{21}, a_{22}, a_{23} \rangle$)

. Solution : points where 2 planes meet $\Rightarrow$ 2 possibilities :

(1)



or



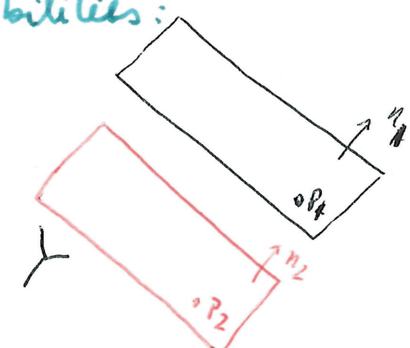
2 planes meet at a line

2 planes coincide

$\underbrace{\hspace{10em}}$

(infinitely many solutions!)

(2)



2 parallel planes
(no solution)

[more unknowns than equations : always have either none or infinitely many solutions]

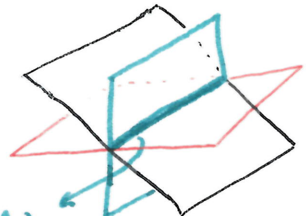
3x3 systems

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

not all coeffs $= 0$

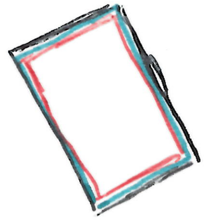
Solution : Points where the 3 planes meet $\Rightarrow$ 3 possibilities

(1) Infinitely many solns



line of solns

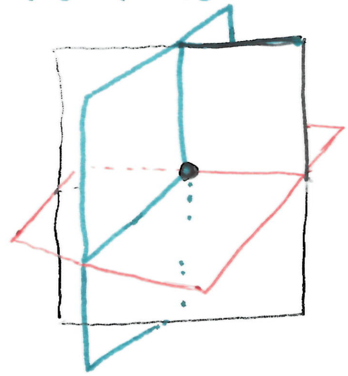
or



→ plane of solns

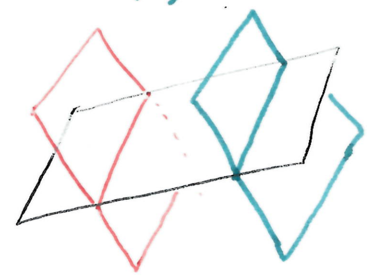
- all 3 planes contain the same line but not all 3 coincide
- 3 planes coincide

(2) One solution



- no 2 planes are parallel
- no 2 planes coincide

(3) No soln



2 parallel planes

⇓

pairwise intersections give 2 parallel lines