Lecture II: §1.1 (cont.) Matrices & §1.2 Echelon form

§1 Matrices:

Recall: $m\times n$ system of linear equations
(x_j) unknowns, (a_{ij}) coefficients, (b_i) constant terms

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases} \quad (**)$$

Why? Matrices give a convenient framework to represent & solve linear systems.

Def An $m\times n$ matrix A is a rectangular array of numbers with m rows & n columns. Write them as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (a_{ij})$$

row # column #

Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ (2x3 matrix) $m=2$, $n=3$; $A = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$ (2x2 matrix) $m=2$, $n=2$ → "square matrix" $m=n$

§2 Matrix Representation of a Linear System:

equations ⇝ rows
unknowns ⇝ columns.
& constant terms

Def: • The coefficient matrix for the system $(**)$ is the $m\times n$ matrix $A=(a_{ij})$

• The augmented matrix ________ is the $m\times(n+1)$ — B, where

$$B = \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{\qquad\qquad}^{A}} & \\ a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right] \begin{array}{l} \\ \leftarrow \text{row } 1 \\ \vdots \\ \leftarrow \text{row } m \end{array} = [A|\underline{b}] \text{ for short}$$

$$\begin{array}{cccc} \uparrow & & \uparrow & \uparrow \\ \text{col } 1 & \cdots & \text{col } n & \text{col } n+1 \end{array}$$

where $\underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

Example:
$$\begin{cases} x_1 + 3x_2 - 2x_3 = 4 \\ 4x_1 - 5x_3 = 0 \\ -2x_1 + 6x_2 = 24 \end{cases}$$

⇝ Align

$$\begin{array}{l} \boxed{1}\, x_1 + \boxed{3}\, x_2 + \boxed{(-2)}\, x_3 = 4 \\ \boxed{4}\, x_1 + \boxed{0}\, x_2 + \boxed{-5}\, x_3 = 0 \\ \boxed{(-2)}\, x_2 + \boxed{6}\, x_2 + \boxed{0}\, x_3 = 24 \end{array}$$

(empty boxes get value $=0$)

$$⇝ B = \left[\begin{array}{ccc|c} 1 & 3 & -2 & 4 \\ 4 & 0 & -5 & 0 \\ -2 & 6 & 0 & 24 \end{array}\right]$$

§3 Elementary operations:

2 steps to solve a system: (1) Reduce the system (i.e. eliminate variables)
(2) Describe the solution set.

$\leadsto$ Want to perform operations on the system to simplify it but keeping the same solution set

Def: 2 systems in n unknowns are equivalent if they have the same solution set.

Theorem 1: The following elementary operations give equivalent systems:

(1) interchange 2 equations (E_i & E_j, write $E_i \longleftrightarrow E_j$)

(2) multiply an equation by a nonzero number & replace it
(E_i, $k \neq 0$, write $E_i \to kE_i$) (can "reverse" it $kE_i \to \frac{1}{k}kE_i = E_i$)

(3) replace an equation by adding to it a constant multiple of a different equation ($E_i \longrightarrow E_i + kE_j$ for $i \neq j$, k any number)
(can "reverse" it $(E_i + kE_j) \downarrow E_i = (E_i + kE_j) + (-k)E_j$)

Example:

$$\begin{cases} x_1 + x_2 = 4 \\ -x_1 + 3x_2 = 9 \end{cases} \xrightarrow[E_1 \to E_1 + E_2]{} \begin{cases} 4x_2 = 13 \\ -x_1 + 3x_2 = 9 \end{cases} \xrightarrow[E_1 \to \frac{1}{4}E_1]{} \begin{cases} x_2 = 13/4 \\ -x_1 + 3x_2 = 9 \end{cases}$$

$$\xrightarrow[E_2 \to E_2 - 3E_1]{} \begin{cases} x_2 = 13/4 \\ -x_1 = 9 - 3 \cdot \frac{13}{4} = \frac{-3}{4} \end{cases} \xrightarrow[E_2 \to (-1)E_2]{} \begin{cases} x_2 = 13/4 \\ x_1 = 3/4 \end{cases} \xrightarrow[E_1 \leftrightarrow E_2]{} \begin{cases} x_1 = 3/4 \\ x_2 = \frac{13}{4} \end{cases}$$

EASY TO SOLVE! (only 1 solution)

By Thm 1, the original system has a unique sol $= \left(\frac{3}{4}, \frac{13}{4}\right)$

§4: On the matrix side = Row operations

Def: There are 3 elementary row operations:

(1) interchange 2 rows ($R_i \longleftrightarrow R_j$)

(2) replace a row by a non-zero scalar multiple of it ($R_i \to kR_i$ for $k \neq 0$)

(3) replace a row by adding to it a constant multiple of a different row ($R_i \to R_i + kR_j$ for $i \neq j$)

Def: Two matrices are row equivalent if we can obtain one from the other by a sequence of elementary row operations.

Note: If these matrices are the augmented matrices of two $(m \times n)$ lin systems then the systems would be equivalent because elementary row operations mimic elementary operations on systems.

GOAL Start from an augmented matrix B, do elem. row operations to get rows with more zeros on a row equiv matrix C, then C is simpler.

Algorithm
(1) from the linear system write the augmented matrix B
(2) do elem. row operations to obtain a simpler matrix C
(3) solve the system that C represent (equivalent to the original one!)

Example:
$$\begin{cases} x_2 + x_3 - x_4 = 3 \\ x_1 + 2x_2 - x_3 + x_4 = 1 \\ -x_1 + x_2 + 7x_3 - x_4 = 0 \end{cases} \rightsquigarrow (1)\ B = \left[\begin{array}{cccc|c} 0 & 1 & 1 & -1 & 3 \\ 1 & 2 & -1 & 1 & 1 \\ \boxed{-1} & 1 & 7 & -1 & 0 \end{array}\right]$$

(2) Row Operations on B: goal = have exactly one 1 in a column & all other entries $=0$ on that column

$$B \xrightarrow[E_3 \to E_3 + E_2]{} \left[\begin{array}{cccc|c} 0 & 1 & 1 & -1 & 3 \\ 1 & 2 & -1 & 1 & 1 \\ 0 & 3 & 6 & 0 & 1 \end{array}\right] \xrightarrow[E_1 \to E_2]{} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & \boxed{3} & 6 & 0 & 1 \end{array}\right]$$

ideal column format, move to the right to repair the next one using 2nd & 3rd

$$\xrightarrow[\substack{E_3 \to E_3 - (3) E_2 \\ [\text{turn } 3 \text{ to } 0]}]{} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & 3 & 3 & -8 \end{array}\right] \xrightarrow[\substack{E_3 \to \frac{1}{3} E_3 \\ [\text{turn } 3 \text{ to } 1]}]{} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & \boxed{1} & -1 & 3 \\ 0 & 0 & 1 & 1 & -8/3 \end{array}\right]$$

(Echelon form example) = staircase

Use 1's to get zeros above them:

$$\xrightarrow[E_2 \to E_2 - E_3]{} \left[\begin{array}{cccc|c} 1 & 2 & \boxed{-1} & 1 & 1 \\ 0 & 1 & 0 & -2 & 17/3 \\ 0 & 0 & 1 & 1 & -8/3 \end{array}\right] \xrightarrow[E_1 \to E_1 + E_3]{} \left[\begin{array}{cccc|c} 1 & \boxed{2} & 0 & 2 & -5/3 \\ 0 & 1 & 0 & -2 & 17/3 \\ 0 & 0 & 1 & 1 & -8/3 \end{array}\right]$$

column is fixed

$$\xrightarrow[E_1 \to E_1 - 2E_2]{} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 6 & -13 \\ 0 & 1 & 0 & -2 & 17/3 \\ 0 & 0 & 1 & 1 & -8/3 \end{array}\right] = C \rightsquigarrow \begin{cases} x_1 + 6x_4 = -13 \\ x_2 - 2x_4 = 17/3 \\ x_3 + x_4 = -8/3 \end{cases}$$

(Reduced Echelon form)

dependent variables

independent variable

$\rightsquigarrow$ Solutions : $\left\{ \begin{array}{l} x_1 = -13 - 6x_4 \\ x_2 = \frac{17}{3} + 2x_4 \\ x_3 = -\frac{8}{3} - x_4 \end{array} \right.$ (general form) for ANY choice of x_4

(*)

How many solutions? $(x_1, x_2, x_3, x_4) = \left(-13-6x_4, \frac{17}{3}+2x_4, -\frac{8}{3}-x_4, x_4\right)$

so infinitely many!

Check that (*) are solutions of the original system:

(Eq 1) $3 \stackrel{?}{=} x_2 + x_3 - x_4 = \frac{17}{3} + 2x_4 + \left(-\frac{8}{3} - x_4\right) - x_4 = \frac{9}{3} = 3$ ✓

(Eq 2) $1 \stackrel{?}{=} x_1 + 2x_2 - x_3 + x_4 = (-13-6x_4) + 2\left(\frac{17}{3}+2x_4\right) - \left(-\frac{8}{3} - x_4\right)$

$+ x_4 = 1 + \underbrace{(-6+4+1+1)}_{=0}x_4 = 1$ ✓

Check that (Eq 3) is also satisfied!

Next time: Echelon form, reduced echelon form & Gauss-Jordan ~~elimi~~ elimination algorithm (to put a matrix in echelon form).