Lecture III : § 1.2 (cont.) Echelon form & Gauss-Jordan elimination

Last Time: ~~Saw~~ linear systems ⟷ augmented matrices $[A|\underline{b}]$

- 3 elementary operations on equations ⟷ 3 elementary operations on rows

$$\begin{cases} (1)\ E_i \longleftrightarrow E_j \\ (2)\ E_i \longrightarrow k E_i \ \text{ for } k \neq 0 \\ (3)\ E_i \longrightarrow E_i + s E_j \ \text{ for } i \neq j \end{cases} \longleftrightarrow \begin{cases} (1)\ R_i \longleftrightarrow R_j \\ (2)\ R_i \longrightarrow k R_i \ \text{ for } k \neq 0 \\ (3)\ R_i \longrightarrow R_i + s R_j \ \text{ for } i \neq j \end{cases}$$

- Systems that are equivalent (same soln. set) ⟷ row-equivalent augmented matrices.
- Staircase form example w/ nice expressions for the solutions

§1 Echelon form:

Def: An $(m \times n)$ matrix B is in echelon form (EF) if

(1) all rows containing only 0's are grouped together at the bottom of B

(2) in every nonzero row, the first nonzero entry (from the left) is a 1.

(3) if a row is nonzero, the first nonzero entry is the the RIGHT of the first nonzero entry of the previous row.

⇝ Staircase form:

Eg:

$$\underset{\text{non REF!}}{\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]} \xrightarrow[E_1 \to E_1 - 4E_2]{} \underset{\text{REF!}}{\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]}$$

both in EF

Def: A matrix B is in reduced echelon form (R.E.F.) if it's in E.F. & the first nonzero entry in any row is the ONLY nonzero entry in its column.

Why? REF matrices give the simplest systems to solve (Example in Lecture II)

Example (1)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 1 \end{array}\right] \longleftrightarrow \begin{cases} x_1 = 5 \\ x_2 + 4x_3 = 7 \\ 0 = 1 \longrightarrow \text{no solution!} \end{cases}$$

(2) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \begin{cases} x_1 = 5 \\ x_2 = 7 - 4x_3 \\ 0 = 0 \end{cases}$ Soln $= (5, 7-4s_3, s_3)$ for any s_3 in $\mathbb{R}$.

dep vars ↳ independent variables. ↳ no info (can ignore it!) ↳ parametric form of a line in $\mathbb{R}^3$

Theorem 1: Given an $(m \times n)$ matrix B, there is a unique $(m \times n)$ matrix C in REF that is row-equivalent to B

Q How to find C? A $\boxed{\text{GAUSS-JORDAN ELIMINATION}}$

STEP 0: If B has all zero entries, done ✓ So B has a nonzero entry.
[Test = 0 Step]

STEP 1: Pick the first (left-most) column with a nonzero entry, say a_{ij}
[find column]

STEP 2: Exchange rows so that this column (j) has a non-zero entry in row one. (so $i=1$)
[swap step]

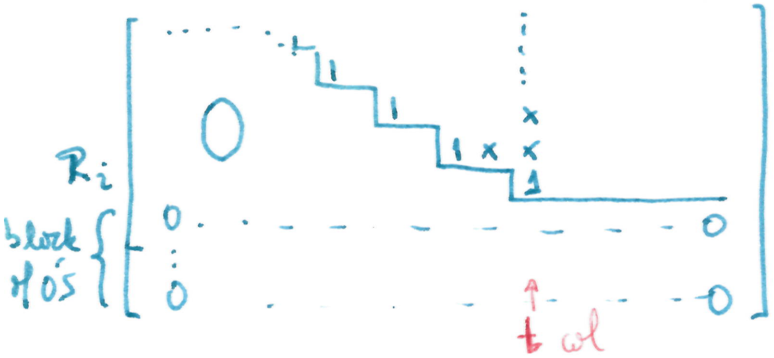
STEP 3: If $a = a_{1j}$ is the first nonzero entry in row 1, use $E_1 \to \frac{1}{a} E_1$ (so now, the first nonzero entry is 1)
[rescale step]

STEP 4: Replace each row R_i ($i \neq 1$) with $R_i - a_{ij} E_1$
[replacement step]

because $(R_i - a_{ij} E_1)_{ij} = a_{ij} - a_{ij} = 0$.

STEPS 5: Work with $\tilde{B}$ (ignore the 1st row of B) & repeat steps 0–4 with $\tilde{B}$ until we get an EF matrix.

STEP 6: Turn EF to REF: Work our way back to put 0's on top of each 1 starting each step.



Start from last non-zero row (R_i) & for each $s < i$ do:

$$R_s \longrightarrow R_s - a_{st} R_i$$

Effect: Entries above row i on col t become 0!

Move 1 row up and repeat to fix the corresp. column.

Example 1: Solve:
$$\begin{cases} x_2 - x_3 + x_4 - x_5 = 1 \\ x_1 - 3x_2 + x_3 - x_4 + x_5 = 3 \\ -2x_2 + 2x_3 \quad + x_4 - x_5 = 2 \\ x_2 - x_3 + 7x_4 - 7x_5 = 9 \end{cases} \longrightarrow \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & -1 & 1 \\ \textcircled{1} & -3 & 1 & -1 & 1 & 3 \\ 0 & -2 & 2 & 1 & -1 & 2 \\ 0 & 1 & -1 & 7 & -7 & 9 \end{array}\right] = B$$

↖ first column w/ a nonzero entry

$$\xrightarrow[\text{(STEP 1)}]{R_1 \to R_2} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & \textcircled{1} & -1 & 1 & -1 & 1 \\ 0 & \boxed{-2} & 2 & 1 & -1 & 2 \\ 0 & 1 & -1 & 7 & -7 & 9 \end{array}\right] \xrightarrow[\text{(STEP 4)}]{R_3 \to R_3 - (-2)R_2} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & \textcircled{1} & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 3 & -3 & 4 \\ 0 & \boxed{1} & -1 & 7 & -7 & 9 \end{array}\right]$$

$$\xrightarrow[\text{(STEP 4)}]{R_4 \to R_4 - R_2} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & \textcircled{3} & -3 & 4 \\ 0 & 0 & 0 & 6 & -6 & 8 \end{array}\right] \xrightarrow[\text{(STEP 2]}]{R_3 \to \frac{1}{3} R_3} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & \textcircled{1} & -1 & 4/3 \\ 0 & 0 & 0 & \boxed{6} & -6 & 8 \end{array}\right]$$

$$\xrightarrow[\text{(STEP 4)}]{R_4 \to R_4 - 6R_3} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 4 \\ 0 & \underset{\uparrow}{0} & 0 & \underset{\uparrow}{0} & 0 & 0 \end{array}\right]$$

Echelon form!

- From EF to REF: only need to treat R_3 & R_2
  $\updownarrow$ column 4, $\updownarrow$ column 2

$$\xrightarrow[\substack{R_1 \to R_1 - (-1)R_3 \\ \text{(STEP 6) for col 4)}}]{R_2 \to R_2 - R_3} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & 0 & 0 & 13/3 \\ 0 & 1 & -1 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & 1 & -1 & 4/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\text{[STEP 6 for col 2]}]{R_1 \to R_1 - (-3)R_2} \left[\begin{array}{ccccc|c} \textcircled{1} & 0 & -2 & 0 & 0 & 10/3 \\ 0 & \textcircled{1} & -1 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & \textcircled{1} & -1 & 4/3 \\ \underset{\uparrow}{0} & \underset{\uparrow}{0} & 0 & \underset{\uparrow}{0} & 0 & 0 \end{array}\right]$$

→ FIXED!

independent variables

Red Echelon form!

$$\begin{cases} \text{indep. variables} = x_3, x_5 \quad \text{(complement)} \\ \text{dependent variables} = x_1, x_2, x_4 \quad (\textcircled{1} \text{ variables}) \end{cases}$$

Set of Solutions: $\begin{cases} x_1 - 2x_3 = 10/3 \\ x_2 - x_3 = -1/3 \\ x_4 - x_5 = 4/3 \end{cases}$ equivalently $\begin{cases} x_1 = \frac{10}{3} + 2x_3 \\ x_2 = \frac{-1}{3} + x_3 \\ x_4 = \frac{4}{3} + x_5 \end{cases}$ for any x_3, x_5

- So general solution has the form $\left(\frac{10}{3} + 2x_3, \frac{-1}{3} + x_3, x_3, \frac{4}{3} + x_5, x_5\right)$
- Particular solutions: (examples) $\cdot\ x_3 = x_5 = 0 \rightsquigarrow \left(\frac{10}{3}, \frac{-1}{3}, 0, \frac{4}{3}, 0\right)$

  $\cdot\ x_3 = \underline{\underline{1}}, x_5 = \underline{-1} \rightsquigarrow \left(\frac{16}{3}, \frac{-4}{3}, \underline{\underline{1}}, \frac{1}{3}, \underline{-1}\right)$

Example 2 Solve $\begin{cases} 2x_1 + 3x_2 - 4x_3 = 3 \\ x_1 - 2x_2 - 2x_3 = -2 \\ -x_1 + 16x_2 + 2x_3 = 16 \end{cases}$ ⇝ Aug. matrix $\left[\begin{array}{ccc|c} \textcircled{2} & 3 & -4 & 3 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 16 \end{array}\right]$

↑ first non-zero column & non-zero entry in row 1

$$\xrightarrow[R_1 \to \frac{1}{2} R_1]{\text{Rescale step}} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 16 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1}]{\text{Replacement Step}} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & \enclose{circle}{-7/2} & 0 & -7/2 \\ 0 & 35/2 & 0 & 35/2 \end{array}\right] \xrightarrow[R_2 \to \frac{-2}{7} R_2]{\text{Rescale}} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 35/2 & 0 & 35/2 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - \frac{35}{2} R_2]{} \left[\begin{array}{ccc|c} 1 & \boxed{3/2} & -2 & 3/2 \\ 0 & \textcircled{1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\substack{\text{EF} \to \text{REF} \\ \text{step.} \\ R_1 \to R_1 - \frac{3}{2} R_2}]{} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(*)

⇝ Equiv. System: $\begin{cases} x_1 - 2x_3 = 0 \\ x_2 = 1 \end{cases}$ ⇝ $\boxed{\begin{matrix} x_1 = 2x_3 \\ x_2 = 1 \end{matrix}}$ general form

$$(x_1, x_2, x_3) = (2\underline{x_3}, 1, \underline{x_3}) = (0, 1, 0) + \underline{x_3}(2, 0, 1)$$ infinitely many solutions

Example 3: Solve $\begin{cases} 2x_1 + 3x_2 - 4x_3 = 3 \\ x_1 - 2x_2 - 2x_3 = -2 \\ -x_1 + 16x_2 + 2x_3 = 16 + \frac{1}{2} \end{cases}$ B = augmented matrix is

row equivalent to $\left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 + \frac{1}{2} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1/2 \end{array}\right]$ REF matrix

(until (*) we do the same row elementary operations as in Example 2)

↪ INCOMPATIBLE system!

NO solutions!

Example 4: Solve $\begin{cases} 2x_1 + 3x_2 - 4x_3 = 3 \\ x_1 - 2x_2 - 2x_3 = -2 \\ -x_1 + 16x_2 + \underline{3}x_3 = 16 \end{cases}$ First two step as in Example 2

System gives an augmented matrix row equivalent to $\left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & \textcircled{1} & 0 & 1 \\ 0 & 35/2 & 1 & 35/2 \end{array}\right]$

$$\xrightarrow[R_3 \to R_3 - \frac{35}{2} R_2]{} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_1 \to R_1 + 2R_3]{} \left[\begin{array}{ccc|c} 1 & 3/2 & 0 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_1 \to R_1 - \frac{3}{2} R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

REF matrix

⇝ system $\begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 0 \end{cases}$ ⇝ Unique Solution = $(0, 1, 0)$.