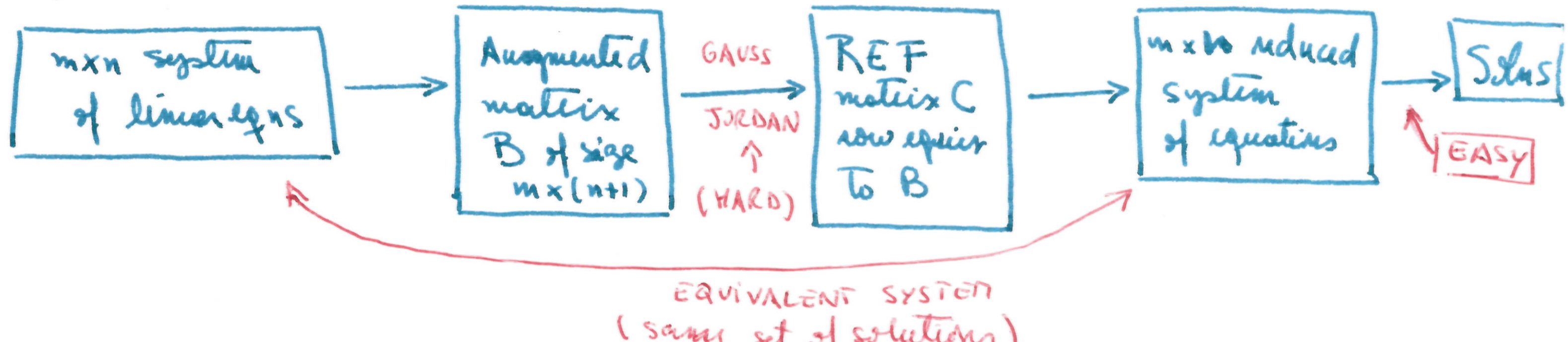
# Lecture IV: §1.3 Consistent systems of linear equations

**Last time**: discussed Gauss-Jordan elimination algorithm : $\underset{(m\times n)}{B} \rightsquigarrow \underset{(m\times n)}{C}$

where $C$ is the UNIQUE matrix in Reduced Echelon Form that is row-equiv to $B$.

Why? It helps solve the associated system by finding an equivalent one




**Def**: A system is consistent when it admits a solution (either 1 or infinitely many)

If no solution exists we call it inconsistent

**Example**: A $2\times 2$ system representing the intersection of 2 lines $L_1$ & $L_2$ in $\mathbb{R}^2$ :


① If the lines are parallel, the system is incompatible

② If not, the lines meet at 1 point (if $L_1 \neq L_2$) or at the entire line (if $L_1 = L_2$)

In both cases, the system is compatible.

## §1 Solution possibilities for a Consistent System

Q: What can we say about the number of solutions without solving the system?

Write: $B = [A\,|\,b] \underset{\text{row equiv}}{\sim} C = [A'\,|\,b']$ in REF (size $= m\times(n+1)$)

Recall: $C = \begin{bmatrix} 1 & & * \\ & 0 & \\ 0 & \cdots & 0 \end{bmatrix}$ contains a row $[0 \ldots\ldots 0\,|\,1]$ (last nonzero row) $\iff$ the system is INCONSISTENT

(examples in Lecture III)

Remark: From REF, on a given nonzero row (say $i$), the first nonzero entry is the only nonzero entry in its column. If that column is $j$, then $x_j$ appears only in Equation $i$.

$\Rightarrow$ We get an expression for $x_i$ in terms of the variables not associated to leading terms

**Example**:

$$C = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 10 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \end{array}$$

$x_1$ only in Eqn 1
$x_2$ " " Eqn 2
$x_4$ " " "

rest of variables ($=\{x_3\}$) can appear in principle in all eqns.

$\Rightarrow$ can write $x_1, x_2, x_4$ in terms of $x_3$ :

$$\begin{cases} x_1 = 10 - 3x_3 \\ x_2 = 1 - 2x_3 \\ x_4 = 4 \end{cases}$$

**Prop**: Every variable corresponding to a leading 1 in C is a dependent variable, that is it can be expressed in terms of the independent ("nonleading-1") vars. The independent variables can have arbitrary value.

**Remark 2**: If C has rows containing only 0's, they don't contribute to the solns. Only the nonzero rows give expressions for the dependent variables.

**Define**: rank(C) = #{non-zero rows in C} (only for REF matrices)

**Note**: rank(C) = #{dependent variables} $\leq$ # n for a compatible system.

If C gives an incompatible system, then rank(C) = #{dependent vars} + 1 $\leq n+1$
(so last nonzero row in C is $[0 \cdots 0 \mid 1]$ ↳ constant term column.)

**Conclusion**: rank(C) $\leq n+1$

If the system is consistent, then rank(C) $\leq n$. & there are $n-\operatorname{rank} C$ independent variables.

**Theorem**: If C is the augmented matrix of a reduced system of size $m \times n$ (so C is in REF) & the system is consistent, then rank(C) $\leq n$ & there are $n-\operatorname{rank}(C)$ independent variables (that can be assigned arbitrary values).
Furthermore, there are infinitely many solutions if and only if rank $< n$

**Corollary**: A linear system has one, infinitely many or no solutions.

Special case: if $m < n$, then the system has either infinitely many or no solutions.

**Examples**: (Handout from Lecture III)

① $C = \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1/2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$

INCONSISTENT

② $\left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

rank C = 3 = n & no $[0 \cdots 0 \mid 1]$ row
⇒ consistent with unique sol.
(0, 1, 0)

③ $C = \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -7/2 & 3/2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

rank C = 2 < n = 3 & no row $[0 \cdots 0 \mid 1]$
so CONSISTENT with infinitely many solutions ($x_1, x_2$ dependent)

$$\begin{cases} x_1 = 3/2 + 7/2\, x_3 \\ x_2 = 0 - x_3 \end{cases}$$ general form

$\underline{x} = \left(\frac{3}{2}+\frac{7}{2}x_3, -x_3, x_3\right)$
$= \left(\frac{3}{2}, 0, 0\right) + \left(\frac{7}{2}x_3, -x_3, x_3\right)$
$= \underbrace{\left(\frac{3}{2}, 0, 0\right)}_{\text{point in } L} + x_3 \underbrace{\left(\frac{7}{2}, -1, 1\right)}_{\text{direction of } L}$ for ANY $x_3$

This is the parametric form of a line L in $\mathbb{R}^3$ (usually with t instead of $x_3$)