Lecture V : 1.3§ (cont) Homogeneous spaces & §1.5 Matrix Operations

Recall: A linear system of equations has either no solution (inconsistent), one solution or infinitely many

If the system has an augmented matrix $B=[A|b]$ in REF, we can decide each case based on $\to [0 \ldots 0|1]$ is a row of C (inconsistent) or not (consistent)

$\searrow$ if consistent, rank (C) = # leading-1 terms $< n \iff$ infinitely many solutions (if and only if)

§1. Homogeneous systems

Def: A system with constant terms $(b_1, \ldots, b_m)$ all $=0$ is called homogeneous. ie $B=[A|\underline{b}]=[A|\underline{0}]$

Note: The system is
$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{cases}$$
so $x_1 = x_2 = \cdots = x_n = 0$ is ALWAYS a solution

We call it the trivial or zero solution.

Theorem: A homogeneous $(m\times n)$ linear system with $m<n$ has infinitely many solutions

Example: Use the example from Lecture III with same coefficient matrix but giving a homogeneous system

Solve
$$\boxed{m=4<5=n} \quad \begin{cases} x_2 - x_3 + x_4 - x_5 = 0 \\ x_1 - 3x_2 + x_3 - x_4 + x_5 = 0 \\ -2x_2 + 2x_3 + x_4 - x_5 = 0 \\ x_2 - x_3 + 7x_4 - 7x_5 = 0 \end{cases} \longrightarrow B = \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & -1 & 0 \\ 1 & -3 & 1 & -1 & 1 & 0 \\ 0 & -2 & 2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 7 & -7 & 0 \end{array}\right]$$

Perform the SAME row operations as in the example from Lecture III.

$\to$ Along the Gauss-Jordan elimination, a homogeneous system will remain homogeneous.

So in our calculations, the constants after | will always be 0.

$\rightsquigarrow$ REF associated to B is
$$C = \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{rank}(C) = 3 < 5 = n$$

- rank $(C) = 3 < 5 \to$ 3 dependent vars x_1, x_2, x_4
- independent variables $= x_3 \ \& \ x_5$

Infinitely many solutions!

$$\begin{cases} x_1 = 2x_3 \\ x_2 = x_3 \\ x_4 = x_5 \end{cases} \quad \text{for ANY } x_3, x_5$$

- Trivial sol for $x_3 = x_5 = 0$.

$\rightsquigarrow$ general soln $= (2x_3, x_3, x_3, x_5, x_5)$
$= (2x_3, x_3, x_3, 0, 0) + (0, 0, 0, x_5, x_5)$
$= x_3(2, 1, 1, 0, 0) + x_5(0, 0, 0, 1, 1)$

§1.5 Matrix Operations:

Scalars = real numbers (future lectures = complex numbers)

Def [Identity of matrices] $A = [a_{ij}]$ an $m \times n$ matrix, $B = [b_{ij}]$ " $r \times s$ "

Then $A = B$ if: • $(m = r)$ & $(n = s)$ [same size]

& • $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$ [same entries]

Eg: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = B$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = B$.

§2 Matrix addition & scalar multiplication:

Def: [Addition] Fix A, B 2 $(m \times n)$ matrices. The sum $A+B$ is an $m \times n$ matrix & $(A+B)_{ij} = A_{ij} + B_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Eg: • $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$ $\rightsquigarrow$ $A+B = \begin{bmatrix} 1+3 & 0+1 \\ 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 0 & 1 \end{bmatrix}$

• $\underset{2 \times 2}{A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$, $\underset{2 \times 3}{B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}$ $\rightsquigarrow$ $A+B$ is <u>NOT</u> defined.

Def [Scalar mult.] Fix A an $(m \times n)$-matrix, r in $\mathbb{R}$ a <u>scalar</u>. The product rA is an $(m \times n)$-matrix & $(rA)_{ij} = r a_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Eg: $r = 2$ $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ $\rightsquigarrow$ $2A = \begin{bmatrix} 2 \cdot 1 & 2 \cdot 2 \\ 2 \cdot 0 & 2 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$.

§3. Vectors in $\mathbb{R}^n$ & vector form of the General Solution

A point in $\mathbb{R}^n$ is an ordered n-tuple of real numbers $\underline{x} = (x_1, \ldots, x_n)$

Def A (column) n-dimensional vector is an $(n \times 1)$-matrix. $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Eg: $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a (column) 3-dimensional vector. Others: $y = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$, $z = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$

Write $\mathbb{R}^n$ to be the set of all (column) n-dimensional vectors. We call it <u>Euclidean n-space</u>.

• Since the elements of $\mathbb{R}^n$ are $n \times 1$ matrices, can add them & multiply by scalars in $\mathbb{R}$.

Use this to write the general solution to a linear system in <u>Vector Form</u>

Example (cont.) Solution is $\begin{cases} x_1 = 2x_3 \\ x_2 = x_3 \\ x_4 = x_5 \end{cases} \longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ x_3 \\ x_3 \\ x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ x_3 \\ x_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_5 \\ x_5 \end{bmatrix}$

$= x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ $\rightsquigarrow$ vector form for the general solution of the homogeneous system = one summand per independent variable (& they are scalars)

Example (non-homogeneous):

$$(*) \quad \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] \rightsquigarrow \begin{cases} x_1 = 1 + 2x_3 \\ x_2 = -3 + x_3 \\ x_4 = x_5 \end{cases}$$

indep vars $\qquad$ x_3, x_5 indep

General soln:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1+2x_3 \\ -3+x_3 \\ x_3 \\ x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_3 \\ x_3 \\ x_3 \\ x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_3 \\ x_3 \\ x_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_5 \\ x_5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \underline{x_3} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \underline{x_5} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad x_3, x_5 \text{ indep vars.}$$

So general form = constant vector + sum of scalar products of indep. variables & constant vectors

Remark: • $x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ is the solution of the homogeneous system associate to (*) $= [\underline{A}|b]$, namely: $\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] = [\underline{\underline{A}}|\underline{0}]$

• $\begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution of the system (*)

Note: This will always be the case!

§4 Scalar product in $\mathbb{R}^n$:

Def Given 2 vectors $\underline{u}, \underline{v}$ in $\mathbb{R}^n$, we define their scalar (or dot) product as the number $\underline{u} \cdot \underline{v} := u_1 v_1 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i$.

Eg: $\underline{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$ $\rightsquigarrow u \cdot v = 1(-1) + 2 \cdot 4 + 3 \cdot 0$
$= -1 + 8 + 0 = \boxed{7}$

Next time: Define matrix multiplication & matrix form of a linear system.