# Lecture VI: §1.5 (cont.) Matrix multiplication, §1.6 Algebraic properties

Recall (Last time)
- defined addition, scalar multiplication on matrices (operate entry by entry)
- defined dot product on vectors in $\mathbb{R}^n = (\underline{n} \times 1)$ matrices
- used this to write the general solution to a system of equations.

**TODAY**: Last operation (multiplication) & algebraic properties of all the operations on $(m \times n)$ matrices

## §1 Matrix multiplication of a matrix A an a vector x

- Only defined if $A$ has size $m \times \underline{n}$ & $x$ is in $\mathbb{R}^{\underline{n}}$
- If so, $Ax$ is a vector in $\mathbb{R}^m$ and

$$\boxed{Ax = x_1(\mathrm{col}_1 A) + x_2(\mathrm{col}_2 A) + \cdots + x_n(\mathrm{col}_n A)}$$

$$(Ax)_{i1} = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = \sum_{j=1}^{n} a_{ij}x_j \quad \text{for } 1 \le i \le m$$

Why? Using this operation, we can write a system with augmented matrix $[A \mid \underline{b}]$ as $Ax = b$ for $A$ an $m \times n$ matrix, $b$ in $\mathbb{R}^m$, $x$ in $\mathbb{R}^n$

**Example 1**: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \rightsquigarrow Ay = \begin{bmatrix} 1\cdot 1 + 2\cdot(-1) + 3\cdot 0 \\ 0\cdot 1 + 1\cdot(-1) + (-1)\cdot 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

In particular $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is a solution of the system $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

**Example 2**: $\underset{2\times 3}{A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}}$, $y = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in \mathbb{R}^2 \rightsquigarrow Ay$ not defined!

**Example 3**: Solve $a\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + b\begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2/3 \\ -1 \end{bmatrix}$ for $a, b$ in $\mathbb{R}$.

By definition $(\mathrm{LHS}) = \begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$ so we have a system $\left[\begin{array}{cc|c} 1 & 4 & 1 \\ 2 & 0 & -2/3 \\ 2 & -1 & -1 \end{array}\right]$

Use Gauss-Jordan elimination:

$$\left[\begin{array}{cc|c} 1 & 4 & 1 \\ 2 & 0 & -2/3 \\ 2 & -1 & -1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 4 & 1 \\ 0 & -8 & -8/3 \\ 0 & -9 & -3 \end{array}\right] \xrightarrow[R_3 \to R_3/-9]{R_2 \to R_2/-8} \left[\begin{array}{cc|c} 1 & 4 & 1 \\ 0 & 1 & 1/3 \\ 0 & 1 & 1/3 \end{array}\right] \xrightarrow[R_1 \to R_1 - 4R_2]{R_3 \to R_3 - R_2} \left[\begin{array}{cc|c} 1 & 0 & -1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{array}\right]$$

so $a = -1/3$, $b = 1/3$.

## §2 Matrix multiplication of 2 matrices:

- Only defined for $A$ of size $m \times \underline{n}$ & $B$ of size $\underline{n} \times s$. $A = (a_{ij})$, $B = (b_{kl})$
- If so, $AB$ is an $(m \times s)$ matrix and $\boxed{\underbrace{j^{\text{th}} \text{ column of } AB}_{\text{vector in } \mathbb{R}^m} = A\,(\underbrace{j^{\text{th}} \text{ column } B}_{\text{vector in } \mathbb{R}^n})}$

More precisely:

$$\boxed{(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}. \quad \text{for } 1 \le i \le m,\ 1 \le j \le s}$$

So to compute $(i,j)$-entry of $AB=C$ we use $\text{row}_i(A)$ & $\text{col}_j(B)$

$$\begin{array}{c} \\ \\ i \\ \\ \end{array}\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ \boxed{a_{i1} \cdots a_{in}} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1s} \\ b_{21} & & b_{2j} & & b_{2s} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & & b_{nj} & & b_{ns} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1s} \\ \vdots & \boxed{c_{ij}} & \\ c_{m1} & \cdots & c_{ms} \end{bmatrix}$$

$$\qquad\qquad\qquad\qquad\qquad j$$

**Example 4** $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}_{2\times 3}$ $\quad B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 4 & 1 & 10 \\ 0 & 5 & 1 & 0 \end{bmatrix}_{3\times 4}$ $\rightsquigarrow$ $\underset{2\times 4}{AB} = \begin{bmatrix} -1 & 2\cdot 4+3\cdot 5 & 1+2+3 & 20 \\ -1 & 1\cdot 4+(-1)5 & 1-1 & 10 \end{bmatrix}$

Ex 1

$= \begin{bmatrix} -1 & 23 & 6 & 20 \\ -1 & -1 & 0 & 10 \end{bmatrix}$

$\rightsquigarrow$ $BA$ not defined ($\underset{4}{\text{cols } B} \neq \underset{2}{\text{rows } A}$)

**Example 5**: Combine $\begin{cases} x_1 = 3y_1 - y_2 + y_3 \\ x_2 = -3y_1 + 5y_2 \end{cases}$ & $\begin{cases} y_1 = -4z_1 + z_3 \\ y_2 = z_2 - z_3 \\ y_3 = 0 \end{cases}$, ie write $\underline{x}$ in terms of $\underline{z}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \& \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} -4 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 3(-4)+0 & -1 & 3+1 \\ (-3)(-4)+0 & 5 & -3-5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

$= \begin{bmatrix} -12 & -1 & 4 \\ 12 & 5 & -8 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

**Conclusion**: $\begin{cases} x_1 = -12z_1 - z_2 + 4z_3 \\ x_2 = 12z_1 + 5z_2 - 8z_3 \end{cases}$

**Example**: Given $A = \begin{bmatrix} -1 & 2 & -39 \\ 1 & 0 & 4 \\ 5 & 1 & 2 \end{bmatrix}$, solve $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ & $Ay = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$

**Solution**: $A \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}$ $\rightsquigarrow$ Use Gauss-Jordan on $\left[\begin{array}{ccc|c|c} -1 & 2 & -39 & 1 & 0 \\ 1 & 0 & 4 & 2 & 1 \\ 5 & 1 & 2 & 3 & 4 \end{array}\right]$

$$\xrightarrow[R_3 \to R_3 - 5R_2]{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|c|c} 0 & 2 & -35 & 3 & 1 \\ 1 & 0 & 4 & 2 & 1 \\ 0 & 1 & -18 & -7 & -1 \end{array}\right] \xrightarrow[R_3 \leftrightarrow R_2]{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c|c} 1 & 0 & 4 & 2 & 1 \\ 0 & 1 & -18 & -7 & -1 \\ 0 & 2 & -35 & 3 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c|c} 1 & 0 & 4 & 2 & 1 \\ 0 & 1 & -18 & -7 & -1 \\ 0 & 0 & 1 & 17 & 3 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - 4R_3]{R_2 \to R_2 + 18R_3} \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & -66 & -11 \\ 0 & 1 & 0 & 299 & 53 \\ 0 & 0 & 1 & 17 & 3 \end{array}\right]$$

so $x = \begin{bmatrix} -66 \\ 299 \\ 17 \end{bmatrix}$ & $y = \begin{bmatrix} -11 \\ 53 \\ 3 \end{bmatrix}$

## §3 Identity matrix:

**Def** For any $n$ in $\mathbb{N}$, the identity matrix of size $n \times n$ has 1's in the diagonal & 0's elsewhere: $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}$ diagonal. Eg $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, etc

## §4 Algebraic properties of matrix operations

**Theorem 1**: $A, B, C$ $(m\times n)$ matrices. Then:

(1) $A+B = B+A$ [Commutative]

(2) $(A+B)+C = A+(B+C)$ [Associative]

(3) There exists a zero matrix $\mathbb{O}$ of size $m\times n$ with $A+\mathbb{O} = \mathbb{O}+A = A$ for any $A$.
$\left[\ \mathbb{O}_{ij} = 0 \text{ for all } \begin{matrix} 1\leq i\leq m \\ 1\leq j\leq n\end{matrix}\right]$

(4) [Inverses] given $A$, there is an $(m\times n)$ matrix $P$ with $A+P = P+A = \mathbb{O}$
$\left[\ P_{ij} = -(A_{ij}) \text{ for all } \begin{matrix} 1\leq i\leq m \\ 1\leq j\leq n\end{matrix}\right]$

Justification: Addition is done on each entry separately & addition in $\mathbb{R}$ satisfies (1) — (4).

**Theorem 2**: $A$ of size $m\times n$, $B$ of size $n\times s$, $C$ of size $s\times q$

(1) $(AB)C = A(BC)$ [Associative]

(2) $\alpha, \beta$ scalars in $\mathbb{R}$ $\alpha(\beta A) = (\alpha\beta)A$.

(3) $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

(4) $I_n$ = identity matrix of size $n\times n$
$I_m$ = identity matrix of size $m\times m$
Then: $A = I_m A = A I_n$

Proof: Use the definitions! In all 3 cases all matrices have the same size, so the identities will follow if we show all entries agree.

(1) $(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} \rightsquigarrow ((AB)C)_{il} = \sum_{j=1}^{n} (AB)_{ij} C_{jl} =$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} A_{ik}B_{kj}C_{jl} \underset{\substack{\uparrow \\ \text{exchange sums}}}{=} \sum_{k=1}^{n}\sum_{j=1}^{n} A_{ik}B_{kj}C_{jl} = \sum_{k=1}^{n} A_{ik} \underbrace{\left(\sum_{j=1}^{n} B_{kj}C_{jl}\right)}_{=(BC)_{kl}} = (A(BC))_{il} \quad \square$$

(2) $(\alpha(\beta A))_{ij} = \alpha(\beta A)_{ij} = \alpha\beta A_{ij} = ((\alpha\beta)A)_{ij}$ $\square$

(3) $(\alpha(AB))_{ij} = \alpha(AB)_{ij} = \alpha\sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{n} A_{ik}\underbrace{\alpha B_{kj}}_{=(\alpha B)_{kj}} = (A(\alpha B))_{ij}$

$= \sum_{k=1}^{n} (\alpha A)_{ik} B_{kj} = ((\alpha A)B)_{ij}$ $\square$

(4) $(I_m A)_{ij} = \sum_{k=1}^{m} (I_m)_{ik}A_{kj} = 1\cdot A_{ij} = A_{ij}$ $\square$

**Theorem 3**: Relate + & multiplication

(1) $A, B$ of size $m\times n$, $C$ of size $(n\times p)$, then $(A+B)C = AC+BC$

(2) $A$ of size $m\times n$, $B, C$ of size $(n\times p)$, then $A(B+C) = AB+AC$

(3) $\alpha, \beta$ scalars, $A$ of size $m\times n$, then $(\alpha+\beta)A = \alpha A+\beta A$

(4) $\alpha$ scalar, $A, B$ of size $m\times n$, then $\alpha(A+B) = \alpha A+\alpha B$.

Proof: Use the definitions! The matrices on each side of $=$ have the same size, so we need to compare the entries of each side to show the desired equalities.

(1) $((A+B)C)_{ij} = \sum_{k=1}^{n} (A+B)_{ik} C_{kj} = \sum_{k=1}^{n} (A_{ik}+B_{ik}) C_{kj} = \sum_{k=1}^{n} A_{ik}C_{kj} + B_{ik}C_{kj}$

$\underset{\text{reorder sum}}{=} \sum_{k=1}^{n} A_{ik} C_{kj} + \sum_{k=1}^{n} B_{ik} C_{kj} = (AC)_{ij} + (BC)_{ij} = (AC+BC)_{ij}$ □

(2) $(A(B+C))_{ij} = \sum_{k=1}^{n} A_{ik} (B+C)_{kj} = \sum_{k=1}^{n} A_{ik}(B_{kj}+C_{kj}) = \sum_{k=1}^{n} A_{ik}B_{kj} + A_{ik}C_{kj}$

$\underset{\text{reorder sum}}{=} \sum_{k=1}^{n} A_{ik} B_{kj} + \sum_{k=1}^{n} A_{ik} C_{kj} = (AB)_{ij} + (AC)_{ij} = (AB+AC)_{ij}$ □

(3) $((\alpha+\beta)A)_{ij} = (\alpha+\beta) A_{ij} = \alpha A_{ij} + \beta A_{ij} = (\alpha A)_{ij} + (\beta A)_{ij} = (\alpha A + \beta A)_{ij}$ □

(4) $(\alpha(A+B))_{ij} = \alpha (A+B)_{ij} = \alpha(A_{ij}+B_{ij}) = \alpha A_{ij} + \alpha B_{ij} = (\alpha A)_{ij} + (\alpha B)_{ij}$

$= (\alpha A + \alpha B)_{ij}$ □

## §5 The transpose of a matrix:

Idea: interchange rows & columns of $A$

**Def** Given $A$ of size $m\times n$, the transpose of $A$ is a matrix $A^T$ of size $n\times m$ with $(A^T)_{ij} = A_{ji}$ for $1\le i\le n$, $1\le j\le m$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \end{bmatrix} \rightsquigarrow A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 0 \end{bmatrix}$

**Def**: $A$ is symmetric if $A^T = A$ (in particular $m=n$, $A$ is a square matrix)

**Theorem 4**: $A, B$ of size $m\times n$, $C$ of size $n\times p$:

(1) $(A+B)^T = A^T + B^T$

(2) $(AC)^T = C^T A^T$

(3) $(A^T)^T = A$

Proof: As usual both matrices in each identity have the same size, so need to compare the entries on each side:

(1) $((A+B)^T)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = (A^T)_{ij} + (B^T)_{ij} = (A^T+B^T)_{ij}$

(2) $(AC)^T_{ij} = (AC)_{ji} = \sum_{k=1}^{n} A_{jk} C_{ki} = \sum_{k=1}^{n} (A^T)_{kj} (C^T)_{ik} = \sum_{k=1}^{n} (C^T)_{ik} (A^T)_{kj}$

$= (C^T A^T)_{ij}$ □

(3) $(A^T)^T_{ij} = (A^T)_{ji} = A_{ij}$ □.

**Property**: If $A$ has size $n\times n$, then $AA^T$ is a symmetric matrix of size $n\times n$

Pf/ $AA^T$ exists and has size $n\times n$ ✓. $(AA^T)^T \underset{(2)}{=} (A^T)^T A^T \underset{(3)}{=} AA^T$ □

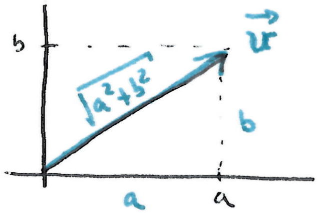
## §6 Scalar products and Vector Norms

Recall: $u$, $v$ in $\mathbb{R}^n$ $\qquad u \cdot v = u_1 v_1 + \dots + u_n v_n = \sum u_i v_i$

Better way $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ $\rightsquigarrow$ $u \cdot v = \underbrace{u^T}_{1 \times n} \underbrace{v}_{n \times 1}$ is an $1 \times 1$ matrix so a number in $\mathbb{R}$.

From Calculus III: Norm of a vector $(v) = \|v\| = \sqrt{v \cdot v} = \sqrt{v^T v}$.

In $\mathbb{R}^2$:




$\vec{v} \cdot \vec{v} = (a, b)\,(a, b) = a^2 + b^2$.

Def: $\|v\| = \sqrt{v_1^2 + \dots + v_n^2} = \sqrt{v^T v}$ is the <u>Euclidean length</u> or <u>Euclidean norm</u> in $\mathbb{R}^n$