Lecture VII: §1.7 Linear independence & Nonsingular matrices

§1. Linear independence:

Def: A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_p\}$ in $\mathbb{R}^m$ is <u>linearly independent</u> (l.i) if the only solution $(a_1, \ldots, a_p)$ to the vector equation $a_1\vec{v}_1 + \cdots + a_p\vec{v}_p = 0$ in $\mathbb{R}^m$ is the trivial one, that is $a_1 = a_2 = \cdots = a_p = 0$.

If a non-trivial solution exists, we say the set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is <u>linearly dependent</u> (l.d)

<u>Example 1</u>: $v_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $v_3 = \begin{bmatrix} 8 \\ 11 \\ 8 \end{bmatrix}$, $v_4 = \begin{bmatrix} 1 \\ +2 \\ 0 \end{bmatrix}$.

- $\{v_1, v_2, v_3\}$ are l.d:

$$a_1\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + a_2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + a_3\begin{bmatrix} 8 \\ 11 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ has a solution $a_1 = -2, a_2 = -1, a_3 = 1$

Q: How to find (a_1, a_2, a_3)? $(\text{LHS}) = \begin{bmatrix} 2 & 4 & 8 \\ 3 & 5 & 11 \\ 1 & 6 & 8 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ & $\begin{bmatrix} 2 & 4 & 8 \\ 3 & 5 & 11 \\ 1 & 6 & 8 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{} \begin{bmatrix} 1 & 6 & 8 \\ 3 & 5 & 11 \\ 2 & 4 & 8 \end{bmatrix}$

$$\xrightarrow[\substack{R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1}]{} \begin{bmatrix} 1 & 6 & 8 \\ 0 & -13 & -13 \\ 0 & -8 & -8 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2/-13 \\ R_3 \to R_3 + 8R_2}]{} \begin{bmatrix} 1 & 6 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 \to R_1 - 6R_2]{} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ so $\begin{cases} a_1 = -2a_3 \\ a_2 = -a_3 \end{cases}$

$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ is a solution for any a_3.

- $\{v_1, v_2, v_4\}$ are l.i:

$$a_1\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + a_2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + a_3\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ has a unique solution (the trivial one)

<u>Why?</u> Use Gauss-Jordan on $\begin{bmatrix} 2 & 4 & 1 \\ 3 & 5 & +2 \\ 1 & 6 & 0 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{} \begin{bmatrix} 1 & 6 & 0 \\ 3 & 5 & +2 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1}]{} \begin{bmatrix} 1 & 6 & 0 \\ 0 & -13 & +2 \\ 0 & -8 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3/-8]{} \begin{bmatrix} 1 & 6 & 0 \\ 0 & -13 & +2 \\ 0 & 1 & -1/8 \end{bmatrix}$

$$\xrightarrow[R_2 \to R_2 + 13R_3]{} \begin{bmatrix} 1 & 6 & 0 \\ 0 & 0 & 3/8 \\ 0 & 1 & -1/8 \end{bmatrix} \xrightarrow[\substack{R_2 \leftrightarrow R_3 \\ R_3 \to \frac{8}{3}R_3}]{} \begin{bmatrix} 1 & 6 & 0 \\ 0 & 1 & -1/8 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 + \frac{1}{8}R_3 \\ R_1 \to R_1 - 6R_2}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ so $\begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{cases}$

<u>Remark</u>: If $\{v_1, \ldots, v_p\}$ are l.d and $a_i \neq 0$, then:

$a_1v_1 + \cdots + a_iv_i + \cdots + a_pv_p = 0$ is equivalent to $\underbrace{a_iv_i}_{\text{in } \mathbb{R}^m} = \underbrace{\sum_{j \neq i} a_jv_j}_{\text{in } \mathbb{R}^m}$

Since $a_i \neq 0$, we multiply both sides by $\frac{1}{a_i}$ and write

$v_i = \sum_{j = i} \left(\frac{a_j}{a_i}\right) v_j$, so v_i can be expressed as a <u>linear combination</u> of the other vectors $\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_p\}$

Conversely, if such an expression exists, then $\{v_1, \dots, v_p\}$ is a l.d. set.

Why study linear independence? $A \cdot x = \mathbb{0}$ in $\mathbb{R}^m$, A of size $m \times n$.

• Prop: The system has a unique solution if and only if the n columns of A are linearly independent vectors.

Proof: $A \cdot x \underset{\substack{\uparrow \\ \text{last time}}}{=} x_1 \operatorname{col}_1 A + \dots + x_n \operatorname{col}_n A = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{R}^m$.

• Prop: The system $A \cdot x = \underline{b}$ for $\underline{b}$ in $\mathbb{R}^m$ admits a solution if and only if $\underline{b}$ is a linear combination of the columns of A.

Proof: Write $A\,x = x_1 \underbrace{\operatorname{col}_1 A}_{\text{in } \mathbb{R}^m} + \dots + x_n \underbrace{\operatorname{col}_n A}_{\text{in } \mathbb{R}^m} = \begin{bmatrix} b_1 \\ b_2 \\ b_m \end{bmatrix}$ for some $x_1, \dots, x_n$ in $\mathbb{R}$ means $\underline{b}$ is a lin comb. of the n columns of A.

Remark: Any set containing the zero vector $\mathbb{0}$ in $\mathbb{R}^m$ is linearly dependent.

Proof For $\{\mathbb{0}, v_2, \dots, v_p\}$ $\quad a_1 \mathbb{0} + a_2 v_2 + \dots + a_p v_p = \mathbb{0}$ has the nontrivial solution $a_1 = 1$, $a_2 = \dots = a_p = 0$.

§2 Unit Vectors:

On $\mathbb{R}^n$ we have n unit vectors (analog of $\check{i}, \check{j}, \check{k}$ in $\mathbb{R}^3$):

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_{n-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Prop: $\{e_1, e_2, \dots, e_n\}$ are linearly independent.

Proof: $x_1 e_1 + \dots + x_n e_n = \mathbb{0}$ if and only if $\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}}_{= I_n \ (n \times n \text{ identity matrix})} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, so

$\mathbb{0} = I_n \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is the only solution! $\square$

Note: Any vector in $\mathbb{R}^n$ is a linear combination of $\{e_1, \dots, e_n\}$, namely

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + x_2 e_2 + \dots + x_n e_n.$$

§3 Useful properties:

Prop 0: Reordering a set of vectors preserves linear independence.

Prop 1: A subset of a linearly independent set is also l.i.

Proof: If $a_1 v_1 + \dots + a_s v_s = \mathbb{0}$ has a non-trivial solution, then $(a_1, \dots, a_s, 0, \dots, 0)$ will be a nontrivial solution of $a_1 v_1 + \dots + a_s v_s + a_{s+1} v_{s+1} + \dots + a_p v_p = \vec{0}$, so the original set was l.d., contradicting our statement. $\square$

Thm 1: Let $\{v_1, \dots, v_p\}$ be a set of vectors in $\mathbb{R}^m$. If $p > m$, then this set is linearly dependent.

Proof: We want to solve $\overset{A}{[v_1 \dots v_p]} \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{R}^m$. If $p > m$,

the $m = \#$ equations $= \#$ rows of A $< \#$ cols of $A = \#$ unknowns $= p$.

Use Gauss-Jordan to write $[A|\mathbb{0}] \underset{\text{row equiv}}{\sim} [A'|\mathbb{0}]$ of A' of size $p \times m$ is REF.

Since a homogeneous system is always compatible,

We have $\#$ dependent unknowns $\overset{\downarrow}{=} \#$ rank $A' \leq \#$ equations $= m < p$

& $p = \#$ unknowns, we conclude $\#$ dep. variables $<$ Total $\#$ vars

So we have at least 1 indep variable, so the system $A\underline{x} = \mathbb{0}$ has infinitely many solutions, in particular a non trivial one.

By definition, the set $\{v_1, \dots, v_p\}$ is then l.d. $\square$.

Note: If $p \leq m$, both situations can happen:

(1) p unit vectors in $\mathbb{R}^m$ are l.i

(2) a set containing $\mathbb{0}$ in $\mathbb{R}^m$ is l.d.

§4 Non-singular matrices:

Def An $(n \times n)$ matrix A is <u>nonsingular</u> if the only solution to $Ax = \mathbb{0}$ in $\mathbb{R}^n$ is the trivial one (that is, the homogeneous system associated to A is a <u>unique</u> solution).

Otherwise, we say A is <u>singular</u>

Example 2. $A = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is nonsingular,

$A = \begin{bmatrix} 2 & 4 & 8 \\ 3 & 5 & 11 \\ 1 & 6 & 8 \end{bmatrix}$ is singular (see Example 1)