Lecture VIII: §1.7 (cont.) Nonsingular matrices, §1.9 Matrix inverses & their prop.

Recall. $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ are l.i. if and only if the system $Ax = \underline{0}$ (for $A = [v_1 \cdots v_p]$ of size $n\times p$) has a unique solution (the trivial soln), ie $x_1 v_1 + \cdots + x_p v_p = \mathbb{0}$ in $\mathbb{R}^n$ has a unique solution

- Defined A of size $n\times n$ is nonsingular if and only if the system $Ax=0$ has a unique solution

§1 Nonsingular matrices: Fix A of size $n\times n$.

Theorem 1: A is nonsingular if and only if its set of n columns is linearly indep. in $\mathbb{R}^n$

Proof: $Ax = x_1 \operatorname{col}_1 A + \cdots + x_n \operatorname{col}_n A = 0$ has a unique solution if and only if $\{\operatorname{col}_1 A, \ldots, \operatorname{col}_n A\}$ is lin. indep in $\mathbb{R}^n$. $\square$

- Why study them? They lead to compatible systems with unique solutions!

Theorem 2 Fix A of size $n\times n$. The equation $A\underline{x} = \underline{b}$ has a unique solution for every $(n\times 1)$ column vector $\underline{b}$ if and only if A is nonsingular.

Proof ($\Rightarrow$) Take $\underline{b} = \begin{bmatrix}0\\0\\ \vdots\\0\end{bmatrix}$ in $\mathbb{R}^n$. By assumption, $Ax = \begin{bmatrix}0\\ \vdots\\0\end{bmatrix}$ has a unique solution, so the cols of A are lin. indep. By Theorem 1, this means that A is nonsingular.

($\Leftarrow$) Pick any b in $\mathbb{R}^n$, and consider the set $\{\operatorname{col}_1 A, \operatorname{col}_2 A, \ldots, \operatorname{col}_n A, b\}$ of $n+1$ vectors in $\mathbb{R}^n$. Since $n+1>n$, by Thm in Lecture VII we know these vectors form a lin. dependent set, hence there is a nontrivial solution $(s_1, \ldots, s_n, s_{n+1})$ to:

$$s_1 \operatorname{col}_1 A + s_2 \operatorname{col}_2 A + \cdots + s_n \operatorname{col}_n A + s_{n+1} b = \mathbb{0} \qquad (*)$$

- If $s_{n+1}=0$, then $(s_1, \ldots, s_n)$ is a nontrivial solution to $s_1 \operatorname{col}_1 A + \cdots + s_n \operatorname{col}_n A = \vec{0}$ so A is singular. This contradicts our assumption!

We conclude from this that $s_{n+1} \neq 0$.

- We multiply $(*)$ by $1/s_{n+1}$ & get

$$b = \frac{-s_1}{s_{n+1}} \operatorname{col}_1 A + \left(\frac{-s_2}{s_{n+1}}\right) \operatorname{col}_2 A + \cdots + \left(\frac{-s_n}{s_{n+1}}\right) \operatorname{col}_n A.$$

$$b = A \begin{bmatrix} -s_1/s_{n+1} \\ \vdots \\ -s_n/s_{n+1}\end{bmatrix}$$

& so the system has a solution! $x = \left(\frac{-s_1}{s_{n+1}}, \ldots, \frac{-s_n}{s_{n+1}}\right)$

• Claim: The solution is unique:

Why? Assume the vectors s, s' in $\mathbb{R}^n$ are solutions of $A\underline{x}=\underline{b}$. We want to show $s=s'$.

Indeed, $A\underline{s} = A\underline{s}' = \underline{b}$, so we take $\mathbb{0} = \underline{b}-\underline{b} = A\underline{s}-A\underline{s}' \underset{\text{Lecture VI}}{=} A(\underline{s}-\underline{s}')$

So $s-s'$ is a vector in $\mathbb{R}^n$ solving $Ax=\mathbb{0}$.

But A is nonsingular, so the system ↗ has a unique solution (the trivial one).

Then $s-s'=\mathbb{0}$, which means $s=s'$. We conclude $Ax=b$ has a unique solution □.

§1.9 Matrix Inverses and their properties:

Recall: $a \neq 0$ in $\mathbb{R}$ $\rightsquigarrow$ $b=\frac{1}{a}$ satisfies $ab=ba=1$, and $1a=a1=a$

"Every nonzero real number is invertible". (b = multiplicative inverse of a) (1 = Neutral element for multiplication)

Want an analog for matrices!

• We have: if A of size $n\times n$, then $I_n = (n\times n)$ identity matrix satisfies

$$A I_n = I_n A = A. \quad \rightsquigarrow I_n = \text{neutral element for matrix multiplication}$$

• What about inverses? Not always have them

Def: A of size $n\times n$. We say A is invertible if we can find a matrix B of size $n\times n$ satisfying $AB = BA = I_n$.

Lemma 1: If it exists, B is unique so we can write it as A^{-1}.

Proof: Assume we have 2 such matrices: B & B'. Then:

$$B = I_n B = (B'A)B \underset{\text{Asso.}}{=} B'(AB) = B' I_n = B' \qquad \square$$

Warning: Not every nonzero square matrix has an inverse!

Example: $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is NOT invertible.

Why? Write $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{\begin{matrix}1\\0\end{matrix}} & \begin{matrix}0\\1\end{matrix} \end{pmatrix} \overset{?}{=} AB = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix}a+2c\\2a+4c\end{matrix}} & \begin{matrix}b+2d\\2b+4d\end{matrix} \end{pmatrix}$

So $\begin{cases} 1 = a+2c \\ 0 = 2a+4c = 2(a+2c) \overset{\text{plug in here}}{=} 2\cdot 1 = 2 \end{cases}$ so $0=2$, a contradiction!

We conclude no B satisfies $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} B = I_2$. | Note: $\mathrm{col}_2 A = 2\,\mathrm{col}_1 A$ so A is singular

• The example hints how to (i) decide if A is invertible or not
(ii) construct A^{-1}, whenever it exists.

§1. Existence of Inverses & Computation:

Theorem 3: A of size $n\times n$ is invertible if and only if A is nonsingular

Proof: Next Time! (Lecture IX)

Algorithm? $AB = I_n$ translates to n systems of n equations in n unknowns.

$$A\underline{x} = \mathrm{col}_1(I_n) = e_1 \quad , \quad \ldots\ldots , \quad A\underline{x} = \mathrm{col}_n(I_n) = e_n \qquad (**)$$

$(\text{soln} = \mathrm{col}_1(B))$ $\qquad$ $(\text{soln} = \mathrm{col}_n(B))$

We can solve these n systems all at once! $[A\,|\,e_1\,|\,e_2\,|\cdots|\,e_n] = [A\,|\,I_n]$

constants for each system

We use Gauss-Jordan to put this augmented matrix in reduced echelon form.

Claim: A invertible, so $[A\,|\,I_n] \underset{\text{row eq}}{\sim} [A'\,|\,C]$ in REF &

A' cannot have a row of 0's (otherwise the systems $(**)$ wouldn't have a unique solution!)

Since A' has size $\overset{(\text{square!})}{n\times n}$ & is in REF, we have no other option but to get $A' = I_n$.

We conclude $\boxed{(A\,|\,I_n) \underset{\text{row-equiv}}{\sim} [I_n\,|\,C]}$ so $AC = I_n$. (because each one of the columns of C solves the n systems one of $(**)$)

Since A is invertible, $A^{-1}(\,AB = AC\,)$ gives $A^{-1}(AB) = A^{-1}(AC)$

$$B = I_n B = (A^{-1}A)B \qquad (A^{-1}A)C = C$$

So $B = C$ is the inverse of A.

Example: $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$ Decide if A is invertible & if so, find A^{-1}!

- cols of A are l.i. $\Rightarrow$ A nonsingular $\underset{\text{Thm}}{\Rightarrow}$ invertible.
- Use algorithm to build A^{-1}:

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \leftrightarrow R_1]{} \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 - 2R_3 \\ R_1 \to R_3 - 3R_3}]{} \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & -3 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - 4R_2]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -4 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$\overset{\shortparallel}{I_3}$ $\qquad$ "A^{-1}"

Check: $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & -4 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

(Next time we'll see that the other product (in the other order) has to give I_3 as well, so only need to check one product!)