Lecture IX: §1.9 Matrix Inverses

Recall: A of size $n\times n$ is *invertible* if we can find a matrix B of size $n\times n$ satisfying $$AB = BA = I_n.$$

Properties: • if it exists, B is unique. We call it A^{-1}.
• Algorithm $[A|I] \underset{\text{row equiv.}}{\sim} [I|C]$ then $C = A^{-1}$.

§1 Applications & Properties:

Prop 1: Assume A of size $n\times n$ is invertible & we want to solve $A\underline{x} = \underline{b}$ in $\mathbb{R}^n$. Then, the system has a unique solution ($x = A^{-1}\underline{b}$.)

Proof: Multiply both sides of the equation by A^{-1}: $A^{-1}(Ax) = A^{-1}b$. But then: $x = I_n x = (A^{-1}A)x = A^{-1}(Ax) = A^{-1}b$ so the solution is unique! $\square$

Thm 1: Let A, C be of size $n\times n$, both invertible. Then:

(1) A^{-1} is invertible & $(A^{-1})^{-1} = A$

(2) AC is invertible and $(AC)^{-1} = C^{-1}A^{-1}$ (like it happened with "transpose")

(3) for $k\neq 0$ scalar, kA is invertible & $(kA)^{-1} = \frac{1}{k}A^{-1}$.

(4) A^T is invertible & $(A^T)^{-1} = (A^{-1})^T$.

(5) I_n is invertible & $(I_n)^{-1} = I_n$.

Proof: (1) By definition $A^{-1}A = AA^{-1} = I_n$ because A is invertible, but then A^{-1} satisfies $A^{-1}B = BA^{-1} = I_n$ for $B = A$, so A^{-1} is invertible. & $(A^{-1})^{-1} = A$.

(2) Use the definition: $(AC)(C^{-1}A^{-1}) \underset{\text{Assoc}}{=} A(CC^{-1})A^{-1} = (AI_n)A^{-1} = AA^{-1} = I_n$ ✓

$(C^{-1}A^{-1})AC \overset{\text{Assoc}}{=} C^{-1}(A^{-1}A)C = C^{-1}(I_nC) = C^{-1}C = I_n$ ✓

(3) $(kA)(\frac{1}{k}A^{-1}) \underset{\substack{\text{scalars \& matrices}\\ \text{commute}}}{=} k(\frac{1}{k}A)A^{-1} \underset{\text{Assoc}}{=} (k\frac{1}{k})(AA^{-1}) = 1\,I_n = I_n$ ✓

$(\frac{1}{k}A^{-1})(kA) = \frac{1}{k}(kA^{-1})A = (\frac{1}{k}k)(A^{-1}A) = I_n$ ✓

(4) $A^T(A^{-1})^T \underset{\text{Transpose rule}}{=} (A^{-1}A)^T = I_n^T = I_n$ ✓

$(A^{-1})^TA^T = (AA^{-1})^T = I_n^T = I_n$ ✓

(5) $I_nI_n = I_n$ $\square$

§3. Existence of Inverses:

Theorem 2: A of size $n\times n$ is invertible if and only if A is nonsingular (cols are l.i.)

To prove this result we need an auxiliary lemma:

Lemma 1: Fix P, and Q of size $n\times n$ & set $R = PQ$.
If P or Q are singular, then so is R.

Proof: Recall (from Lecture VIII) A singular if and only if $A\underline{x} = \underline{0}$ has a nontrivial sol

- Assume Q is singular & pick $\underline{x}$ with $Q\underline{x} = \underline{0}$, $\underline{x}$ in $\mathbb{R}^n$ nonzero vector.
Then $R\underline{x} = (PQ)\underline{x} = P(Q\underline{x}) = P\,\underline{0} = \underline{0}$ & $\underline{x}\neq\underline{0}$, so R is singular
- Assume Q is nonsingular but P is singular. Then, we can find $\underline{b}$ in $\mathbb{R}^n$ nonzero vector with $P\underline{b} = \underline{0}$.
From Lecture VIII we know that $Q\cdot\underline{x} = \underline{b}$ has a unique solution because Q is nonsingular. Since $\underline{b}\neq\underline{0}$ we know that $\underline{x}\neq\underline{0}$.
But then $R\underline{x} = (PQ)\underline{x} = P(Q\underline{x}) = P\underline{b} = \underline{0}$ & $\underline{x}\neq\underline{0}$ so R is singular. □

Proof of Thm 2:

($\Rightarrow$) We assume A is invertible and write B with $AB = I$.
Since the unit vectors are l.i., I_n is nonsingular, so by Lemma 1: both A & B are nonsingular.

($\Leftarrow$) We assume A is nonsingular. We know that all n systems $A\underline{x} = e_i$ $i=1,\ldots,n$ each admit a unique solution, so we can find B with $\boxed{AB = I_n}$ (*)
To conclude, we must show $BA = I_n$.

Useful trick: $AB = I_n$ is nonsingular, so by Lemma 1 B is nonsingular
The previous argument ensures we can find C with $BC = I_n$.
Then $A = AI_n = A(BC) = (AB)C = I_nC = C$

We get $\boxed{BA = BC = I_n}$ (**)

Combining (*) & (**) we get that A is invertible. □

§2 Special case : 2x2 matrices

Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & define $\Delta = ad - bc$ (determinant)

Prop 2: (1) If $\Delta = 0$, then A does not admit an inverse

(2) If $\Delta \neq 0$, then A is invertible & $A^{-1} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Proof: For (2) we check the definition :

- $A \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{bmatrix} = I_2$ ✓

- $\frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} A = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} da-bc & 0 \\ 0 & -bc+ad \end{bmatrix} = I_2$ ✓

For (1) we try to solve the system $A\,\underline{B} = I_2$ $\quad B = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$.

<u>CASE 1</u>: Assume $b \neq 0$ so $\Delta = ad - bc = 0$ means $c = \frac{ad}{b}$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - \frac{a d}{b} R_1]{} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ \boxed{0 \quad 0} & & -\frac{d}{b} & 1 \end{array}\right]$$

so the system $Ay = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is inconsistent! so A is <u>not</u> invertible

<u>CASE 2</u>: $b = 0$, so $\Delta = ad - 0 = ad = 0$ means either $a = 0$ or $d = 0$

- If $a = 0$, then $A = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$ & $A\,X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ has no solution because $Ax = \begin{bmatrix} 0 \\ cx_1 + dx_2 \end{bmatrix}$

- If $d = 0$, & $a \neq 0$ then $A = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \rightsquigarrow \left[\begin{array}{cc|cc} a & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1 \to \frac{1}{a} R_1]{} \left[\begin{array}{cc|cc} 1 & 0 & 1/a & 0 \\ c & 0 & 0 & 1 \end{array}\right]$

$$\xrightarrow[R_2 \to R_2 - cR_1]{} \left[\begin{array}{cc|cc} 1 & 0 & 1/a & 0 \\ \boxed{0 \quad 0} & & -c/a & 1 \end{array}\right]$$

So the system $Ay = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is inconsistent! □

- This Prop makes the computation of inverses (& decidability!) very easy for $n=2$.

Later in the semester, we'll find a similar result for $n > 2$.

<u>Example</u>: $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $\quad \Delta = 1\cdot 4 - 2\cdot 3 = 4 - 6 = -2 \neq 0$ so A is invertible

& $A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}$

Compare with $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_1]{} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2/-2]{} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -1/2 \end{array}\right] \xrightarrow[R_1 \to R_1 - 3R_2]{} \left[\begin{array}{cc|cc} 1 & 0 & -2 & +3/2 \\ 0 & 1 & 1 & -1/2 \end{array}\right]$

$\overset{\shortparallel}{A^{-1}}$

Summarizing the results from Lecture VIII & today:

Thm 3: Fix A of size $n \times n$. The following statements are equivalent:

(1) A is nonsingular (the only solution to $A\underline{x} = \underline{0}$ is the trivial one)

(2) The columns of A form a linearly independent set

(3) $A\underline{x} = \underline{b}$ has ALWAYS a unique solution (for any choice of $\underline{b}$)

(4) A is invertible

(5) A is row equivalent to I_n.