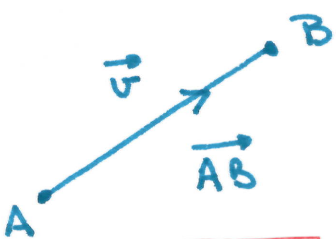
Lecture X : §2.1 Vectors in the Plane, §2.2 Vectors in Space

- From §1.5, we know a vector in $\mathbb{R}^n$ is an $(n\times 1)$ matrix.
- TODAY: Use vectors to model physical quantities (eg forces, displacements, velocities).
  $\rightsquigarrow$ Need a geometric way of representing vectors.

§1 Vectors:

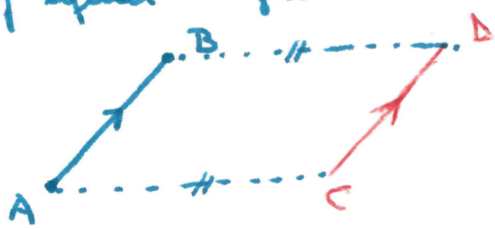
Def: Informally, a vector represents the data of magnitude (or length) and direction. We represent this by a directed line segment:

- length of the segment = magnitude
- arrow : indicates the direction



- initial pt $(\vec{v}) = A$
- terminal pt $(\vec{v}) = B$

(1) We do not distinguish between parallel vectors of equal magnitudes:



$\overrightarrow{AB}$ & $\overrightarrow{CD}$ represent the same vector.

(2) $\vec{0}$ is the unique vector of magnitude 0 (and no direction!)

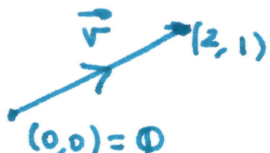
Notation: $|\vec{v}|$ = magnitude of $\vec{v}$

§2 Position Vectors & components of a vector:

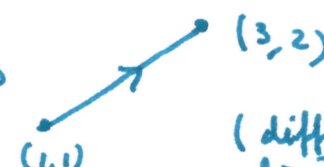
To have a consistent (& algebraic!) representation of vectors, we fix the initial point (tail) of any vector $\vec{v}$ to be the origin O. This is the position vector for $\vec{v}$

Def: head of $\vec{v}$ = pt P, so the position vector is $\overrightarrow{OP}$

- Eg in $\mathbb{R}^2$:

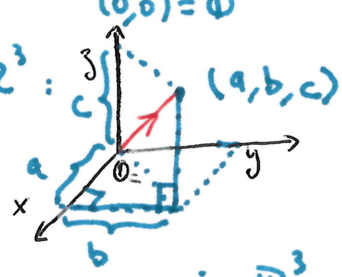


$\rightsquigarrow \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ same as



(different location!)

- Eg in $\mathbb{R}^3$:



$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$O = (0,0,0)$ in $\mathbb{R}^3$

Notice the right angles in the picture

(2) Scalar multiplication:

If $\vec{v}$ is a vector & c is a scalar (in $\mathbb{R}$), then $c\vec{v}$ is a vector determined by

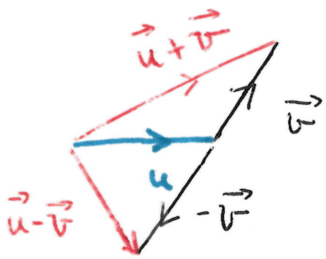
- magnitude of $c\vec{v} = \underbrace{|c|}_{\geq 0} \cdot \|\vec{v}\|$
- direction of $c\vec{v} = \begin{cases} \text{direction of } \vec{v} & \text{if } c>0 \\ \text{opposite to the direction of } \vec{v} & \text{if } c<0 \\ \text{no direction } (c\vec{v}=\mathbb{0}) & \text{if } c=0 \end{cases}$

Note: $\vec{v}$ & $c\vec{v}$ are parallel vectors!

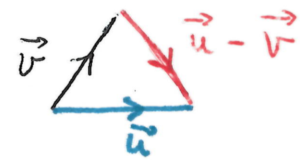
Eg:

(3) Subtraction (difference): Use (1) + (2)!

If $\vec{u}$ & $\vec{v}$ are vectors : $\vec{u}-\vec{v} = \vec{u} + (-\vec{v})$



Directly:



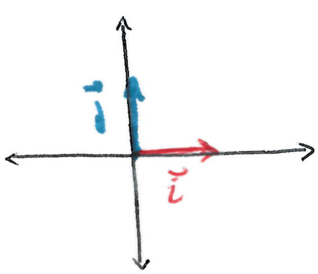
(from head of $\vec{v}$ to head of $\vec{u}$)

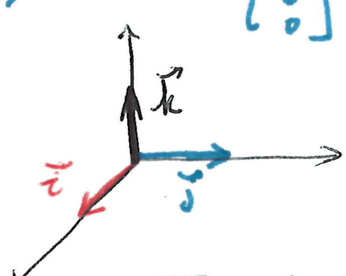
§4 (Basic) Unit vectors

Def: A unit vector is any vector of length $= 1$.

Earlier in the course: we saw canonical (basic) unit vectors $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \to i^{th}$ word.

- In $\mathbb{R}^2$: Write them as $\vec{i}, \vec{j}$, $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- In $\mathbb{R}^3$: ______ $\vec{i}, \vec{j}, \vec{k}$ $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$





Prop: Any vector is a linear combination of basic unit vectors

In $\mathbb{R}^2$ $= \begin{bmatrix} a \\ b \end{bmatrix} = a\vec{i} + b\vec{j}$ In $\mathbb{R}^3$: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\vec{i} + b\vec{j} + c\vec{k}$

Def $\overrightarrow{AB}$ a vector with position vector $v = \overrightarrow{OP}$. The coordinates of P are called <u>components</u> of v.

- <u>In $\mathbb{R}^2$</u>: 2 components : $\begin{cases} x\text{-component} = b_1 - a_1 \\ y\text{-component} = b_2 - a_2 \end{cases}$ if $A = (a_1, a_2)$ & $B = (b_1, b_2)$

- [In the example : $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ components $= \begin{cases} x\text{-comp} : & 3-1 \\ y\text{-comp} : & 2-1 \end{cases}$]

- <u>In $\mathbb{R}^3$</u> 3 components : $\begin{cases} x\text{-comp} = b_1 - a_1 \\ y\text{-comp} = b_2 - a_2 \\ z\text{-comp} = b_3 - a_3 \end{cases}$

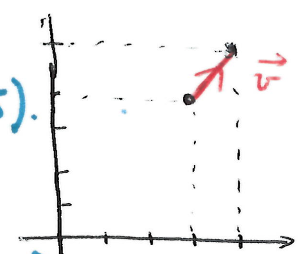
  $A = (a_1, a_2, a_3)$
  $B = (b_1, b_2, b_3)$

Proposition: Two position vectors are equal if and only if their components are the same.

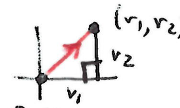
Example: Given $v = \overrightarrow{AB}$ with position vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find the coordinates of B given $A = (3,4)$ & draw $\vec{v}$.

<u>Solution</u>: $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} b_1 - 3 \\ b_2 - 4 \end{bmatrix}$ so $B = (1+3, 1+4) = (4,5)$.

$B = (b_1, b_2)$



- <u>Magnitude</u> $= \|\vec{v}\| = \begin{cases} \sqrt{v_1^2 + v_2^2 + v_3^2} & \text{if } \vec{v} \text{ in } \mathbb{R}^3 \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \text{ (see page 4)} \\ \sqrt{v_1^2 + v_2^2} & \text{if } \vec{v} \text{ in } \mathbb{R}^2 \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \end{cases}$



Pythagorean Thm!

§3 <u>Operations for vectors</u>: Addition, scalar multiplication.

- Algebraically, we do it as for $n \times 1$ matrices, that is, componentwise! ⇝ Algebraic properties hold!
- <u>Geometrically</u> (easier to draw in $\mathbb{R}^2$)

(1) <u>Addition</u> 2 ways

[Triangle Law] Place $\vec{v}$ at the head of $\vec{u}$ & draw the Δ.

[Parallelogram Law] Place two initial pts together (at $\oplus$) draw the ▱ & $\vec{u} + \vec{v}$ = diagonal containing $\oplus$.

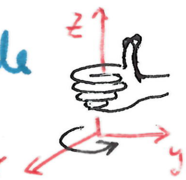
Eg: Find all vectors of length 8 parallel to $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Soln: • Find 2 unit vectors parallel to $\vec{v}$: $\pm \frac{\vec{v}}{\|\vec{v}\|} = \pm \frac{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}{5} = \pm \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

• Scale by 8 $\Rightarrow$ 2 parallel vectors : $\vec{w} = \begin{bmatrix} 24/5 \\ 32/5 \end{bmatrix}$, $\vec{w}' = \begin{bmatrix} -24/5 \\ -32/5 \end{bmatrix}$

§5 Rectangular Coordinates in $\mathbb{R}^3$:

The 3 coordinate axes are directed according to the right hand rule

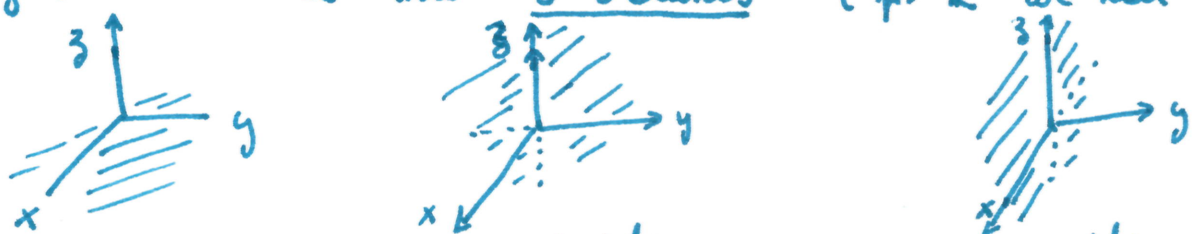


• 3 coordinate planes : (i) xy-plane : (contains x- and y-axes; equation : $z = 0$)

(ii) xz-plane : (contains x- and z-axes ; eqn: $y = 0$)

(iii) yz-plane : contains y- and z-axes, eqn. $x = 0$)

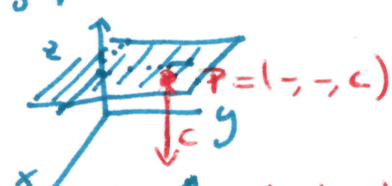
They subdivide $\mathbb{R}^3$ into 8 octants (for $\mathbb{R}^2$ we had 4 quadrants)



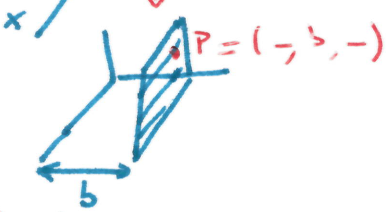
xy-plane y-z-plane xz-plane

Parallel planes to these here equations (i) $z = c$

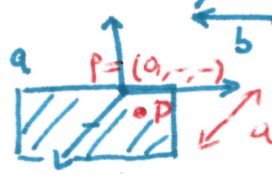
$\rightsquigarrow$ translate the plane so that it passes through a point (a, b, c)



(ii) $y = b$



(i) $x = a$



• Distance from P to Q : $d(P,Q) = \|\overrightarrow{PQ}\|$

Why? Pythagorean Theorem . (*)

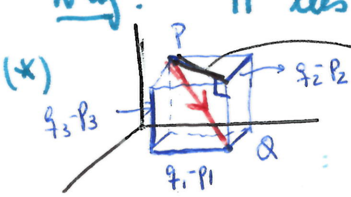
Here: – represent any value we want to put .

§6 Midpoint Formula / Distance Formula

Thm : $P = (p_1, p_2, p_3)$, $Q = (q_1, q_2, q_3)$ The midpoint M of the line segment joining P & Q equals $M = \left(\frac{p_1+q_1}{2}, \frac{p_2+q_2}{2}, \frac{p_3+q_3}{2}\right)$

Why? M lies on the segment & $d(P, M) = d(Q, M)$. $\square$

(*)



$a = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$

So $d(P,Q) = \sqrt{(q_3 - p_3)^2 + a^2} = \|\overrightarrow{PQ}\|$