Lecture XI: §2.3 The Dot & Cross Products

§1 Definition of Dot Product:

Def: $\vec{u}, \vec{v}$ vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, then $\vec{u} \bullet \vec{v} = \underset{1\times n}{u^T}\ \underset{n\times 1}{v}$ (a real number)
(product of a $1\times n$ times an $n \times 1$ matrix, where $n = 2$ or 3)
is called the dot product of $\vec{u}$ and $\vec{v}$

Eg: $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$ $\vec{u} \cdot \vec{v} = [1\ -1\ 1]\begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} = 0 - 3 + 5 = \boxed{2}$.

Algebraic Properties → inherited from matrix operations

For $\vec{u}, \vec{v}, \vec{w}$ vectors & α in $\mathbb{R}$ scalar:

(1) $\vec{u} \bullet \vec{v} = \vec{v} \cdot \vec{u}$ [Commutative]

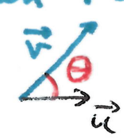
(2) $(\vec{u} + \vec{v}) \bullet \vec{w} = \vec{u} \bullet \vec{w} + \vec{v} \bullet \vec{w}$ [Distributive]

(3) $(\alpha \vec{u}) \bullet \vec{v} = \vec{u} \bullet (\alpha \vec{v}) = \alpha (\vec{u} \bullet \vec{v})$ [Associative]

(4) $\vec{v} \bullet \vec{v} = \|\vec{v}\|^2$.

Geometric form of the dot product:

Fix 2 vectors $\vec{u}, \vec{v}$ & let θ be the angle between them (so $0 \leq \theta \leq \pi$) and $\|\vec{u}\| = a$, $\|\vec{v}\| = b$.

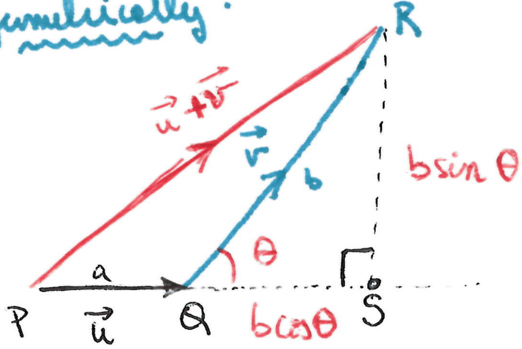


We compute $\|\vec{u} + \vec{v}\|^2$ in two different ways:

• Algebraically $= \|\vec{u} + \vec{v}\|^2 \overset{(4)}{=} (\vec{u} + \vec{v}) \bullet (\vec{u} + \vec{v}) \overset{(2)}{=} \vec{u} \bullet \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}$

$$\underset{(1)+(4)}{=} \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 = \boxed{a^2 + b^2 + 2\vec{u} \cdot \vec{v}} \quad (I)$$

• Geometrically:



$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= |PR|^2 \\ &= |PS|^2 + |RS|^2 \\ &= (a + b\cos\theta)^2 + (b\sin\theta)^2 \\ &= a^2 + 2ab\cos\theta + b^2\cos^2\theta \\ &\qquad + b^2\sin^2\theta \quad \text{sum to 1} \\ &= \boxed{a^2 + 2ab\cos\theta + b^2} \quad (II) \end{aligned}$$

Comparing (I) and (II) we get $\vec{u}\cdot\vec{v} = ab\cos\theta$

Conclusion: $\boxed{\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta}$ | for vectors in $\mathbb{R}^2$ & $\mathbb{R}^3$

We can use this identity to compute the angle between 2 vectors:

Example: ① Assume $\vec{u}$ & $\vec{v}$ have lengths 3 & 6 respectively, and the angle between them is 60°, then:

$$\vec{u}\cdot\vec{v} = 3\cdot 6\cos 60^\circ = \frac{18}{2} = \boxed{9}$$

② Find the angle between $\begin{bmatrix}2\\1\\0\end{bmatrix}$ & $\begin{bmatrix}-1\\1\\5\end{bmatrix}$.

Soln: $$\cos\theta = \frac{[2,1,0]\begin{bmatrix}-1\\1\\5\end{bmatrix}}{\left\|\begin{bmatrix}2\\1\\0\end{bmatrix}\right\|\,\left\|\begin{bmatrix}-1\\1\\5\end{bmatrix}\right\|} = \frac{-1}{\sqrt{5}\sqrt{27}} = \frac{-1}{3\sqrt{15}} \approx 95^\circ.$$

Consequence: Two vectors are perpendicular (or orthogonal), that is, the angle between them is 90°, if and only if their dot product is zero

In symbols: $\vec{u}\perp\vec{v}$ if and only if $\vec{u}\cdot\vec{v}=0$.

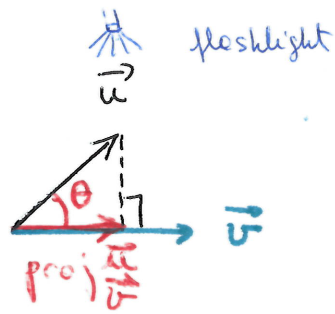
§2. Projections:

Fix $\vec{u}, \vec{v}$ nonzero vectors. We define 2 notions:

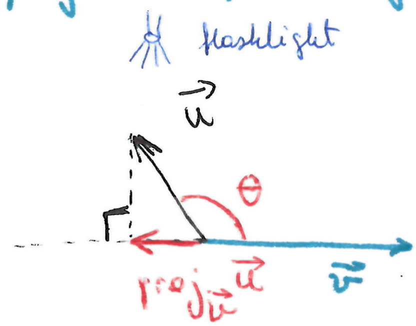
- (orthogonal) projection of $\vec{u}$ onto $\vec{v}$ = vector projection of $\vec{u}$ along $\vec{v}$

$\text{proj}_{\vec{v}}\vec{u}$:

Pictorially :



$0\le\theta\le\frac{\pi}{2}$



$\frac{\pi}{2}<\theta\le\pi$

- signed magnitude of $\text{proj}_{\vec{v}}\vec{u}$ = scalar component of $\vec{u}$ in the direction of $\vec{v}$.

IDEA: $\text{proj}_{\vec{v}}\vec{u}$ is a vector parallel to $\vec{v}$. The signed magnitude indicates if its direction agrees (+) or not (−) with the direction of $\vec{v}$

sign = + if $0\le\theta\le\frac{\pi}{2}$, sign = − if $\frac{\pi}{2}<\theta\le\pi$

Relation: $\quad \text{proj}_{\vec{v}}\vec{u} = \text{comp}_{\vec{v}}\vec{u} \cdot$ unit vector along $\vec{v}$

$$= \underbrace{\boxed{\text{comp}_{\vec{v}}\vec{u}}}_{\text{signed length}} \ \underbrace{\frac{\vec{v}}{\|\vec{v}\|}}_{\text{direction}}$$

Formula for $\text{comp}_{\vec{v}}\vec{u}$: use right-angled triangle in the 2 figures.

$$\text{comp}_{\vec{v}}\vec{u} = |\vec{u}| \cos\theta = |\vec{u}| \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$$

Then $\boxed{\text{proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\,\vec{v}}$

Application: $\quad \vec{u} = \underbrace{(\text{proj}_{\vec{v}}\vec{u})}_{\text{parallel to } \vec{v}} + \underbrace{(\vec{u} - \text{proj}_{\vec{v}}\vec{u})}_{\text{perpendicular to } \vec{v}}$ This is the ONLY way to express $\vec{u} = \vec{w} + \vec{s}$ where $\vec{w} \parallel \vec{v}$, $\vec{s} \perp \vec{v}$

Example: Compute the projections of $\vec{i}, \vec{j}, \vec{k}$ along $\begin{bmatrix}1\\-1\\2\end{bmatrix}$

$$\text{proj}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}}\vec{i} = \frac{\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\-1\\2\end{bmatrix}}{\left\|\begin{bmatrix}1\\-1\\2\end{bmatrix}\right\|^2}\begin{bmatrix}1\\-1\\2\end{bmatrix} = \frac{1}{6}\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

Similarly, $\text{proj}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}}\vec{j} = -\frac{1}{6}\begin{bmatrix}1\\-1\\2\end{bmatrix}$, $\quad \text{proj}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}}\vec{k} = \frac{2}{6}\begin{bmatrix}1\\-1\\2\end{bmatrix}$

Note:
$\text{comp}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}}\vec{i} = \frac{1}{\sqrt{6}}$, $\quad \text{comp}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}}\vec{j} = \frac{-1}{\sqrt{6}}$, $\quad \text{comp}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}}\vec{k} = \frac{2}{\sqrt{6}}$

§3. Cross Product: Only in $\mathbb{R}^3$

Def: Fix $\vec{u} = \begin{bmatrix}a_1\\b_1\\c_1\end{bmatrix}$, $\vec{v} = \begin{bmatrix}a_2\\b_2\\c_2\end{bmatrix}$ two vectors in $\mathbb{R}^3$

The cross product of $\vec{u}$ and $\vec{v}$ is a vector in $\mathbb{R}^3$, defined as

$$\vec{u} \times \vec{v} = \begin{bmatrix} b_1c_2 - b_2c_1 \\ -(a_1c_2 - a_2c_1) \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

- Remember the formula using determinants:

- Recall determinant of a 2×2 matrix $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2$

- determinant of a 3×3 matrix:

$$\begin{vmatrix} x & y & z \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = x \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - y \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + z \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

With these definitions in mind:

$$\vec{u} \times \vec{v} = \begin{array}{c} \\ \vec{u} \\ \vec{v} \end{array} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \underbrace{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}_{\text{scalar}} \vec{i} - \underbrace{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}_{\text{scalar}} \vec{j} + \underbrace{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}_{\text{scalar}} \vec{k}$$

Example: $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix} = \vec{i}\left(2 \cdot 2 - (-1)3\right) - \left(1 \cdot 2 - 2 \cdot 3\right)\vec{j} + \left(1(-1) - 2 \cdot 2\right)\vec{k}$$

$$= 7\vec{i} - (-4)\vec{j} + (-5)\vec{k} = \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}.$$

Properties $\vec{u}, \vec{v}, \vec{w}$ vectors in $\mathbb{R}^3$, α, β scalars

(1) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$ (ANTI commutative) $\rightsquigarrow$ $\vec{u} \times \vec{u} = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for all $\vec{u}$.

(2) $\alpha \vec{u} \times \beta \vec{v} = \alpha\beta (\vec{u} \times \vec{v})$ (Associative) (so $\vec{0} \times \vec{v} = \vec{0}$ for all $\vec{v}$)

(3) $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$ (Distributive)

(4) [Follows from (1) & (3)] $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

Proof (1) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}$ so it follows from the definition.

Rest follow easily

(5) $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

Key Proposition: The vector $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ & $\vec{v}$

Proof: We need to verify $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$ & $\vec{v} \cdot (\vec{u} \times \vec{v})$

By (5) . $\vec{u} \cdot (\vec{u} \times \vec{v}) = \underbrace{(\vec{u} \times \vec{u})}_{\vec{0} \text{ (by (1))}} \cdot \vec{v} = \vec{0} \cdot \vec{v} = 0$ ✓

. $\vec{v} \cdot (\vec{u} \times \vec{v}) \underset{(1)}{=} -\left(\vec{v} \cdot (\vec{v} \times \vec{u})\right) = -\underbrace{(\vec{v} \times \vec{v})}_{\vec{0}} \cdot \vec{u} = 0$ ✓