Lecture 12: §2.3 Cross Product, §2.4 Lines & Planes

§0 **Recall:** $\vec{u}, \vec{v}$ in $\mathbb{R}^3$ $\rightsquigarrow$ $\vec{u}\times\vec{v}$ is a vector in $\mathbb{R}^3$ given as $\det\begin{pmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3\end{pmatrix}$

Key: $\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ & $\vec{v}$

Why? $\cdot$ $\vec{u}\cdot(\vec{u}\times\vec{v}) = (\vec{u}\times\vec{u})\cdot\vec{v} = \vec{0}\cdot\vec{v} = \vec{0}$ ✓

$\cdot$ $\vec{v}\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(-\vec{v}\times\vec{u}) = -(\vec{v}\cdot(\vec{v}\times\vec{u})) = -\vec{0} = \vec{0}$ ✓

- Direction of $\vec{u}\times\vec{v}$ is given by Right hand rule
(or has no direction if $\vec{u}\times\vec{v} = \vec{0}$)

Theorem (Geometric Side): Assume $\vec{u}, \vec{v}$ are nonzero vectors in $\mathbb{R}^3$, and let θ be the angle between $\vec{u}$ & $\vec{v}$ (so $0\leq\theta\leq\pi$). Then

$$\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin\theta$$

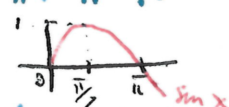
Proof: Write $\vec{u} = \begin{bmatrix} a_1\\ a_2\\ a_3\end{bmatrix}$, $\vec{v} = \begin{bmatrix} b_1\\ b_2\\ b_3\end{bmatrix}$

- Verify that $\|\vec{u}\times\vec{v}\|^2 + (\vec{u}\cdot\vec{v})^2 = \|\vec{u}\|^2\|\vec{v}\|^2$ (exercise)

Then, since $\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$, we get

$$\|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - \|\vec{u}\|^2\|\vec{v}\|^2\cos^2\theta = \|\vec{u}\|^2\|\vec{v}\|^2\underbrace{(1-\cos^2\theta)}_{=\sin^2\theta}$$

We take $\sqrt{\ }$ to conclude $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\,|\sin\theta| \underset{\uparrow}{=} \|\vec{u}\|\,\|\vec{v}\|\sin\theta$. □

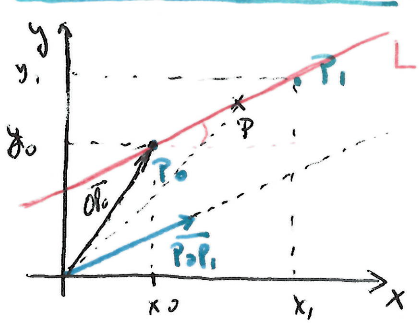


$0\leq\sin\theta$ for $0<\theta\leq\pi$

Conclusion: (1) We know the magnitude & direction of $\vec{u}\times\vec{v}$ from $\vec{u}$ & $\vec{v}$.

(2) $\vec{u}$ & $\vec{v}$ are nonzero parallel vectors (so $\theta = 0$ or π) if and only if $\vec{u}\times\vec{v} = \vec{0}$.

§1 Lines in $\mathbb{R}^2$ & $\mathbb{R}^3$:

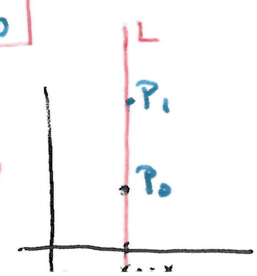


$P_0 = (x_0, y_0)$

$P_1 = (x_1, y_1)$

- Assume $x_0 \neq x_1$: The equation of the line L through P_0 & P_1 is $\boxed{y = m(x - x_0) + y_0}$ with $\boxed{m = \text{slope} = \frac{y_1 - y_0}{x_1 - x_0}}$

- If $x_0 = x_1$, the line is vertical & has equation $\boxed{x = x_0}$

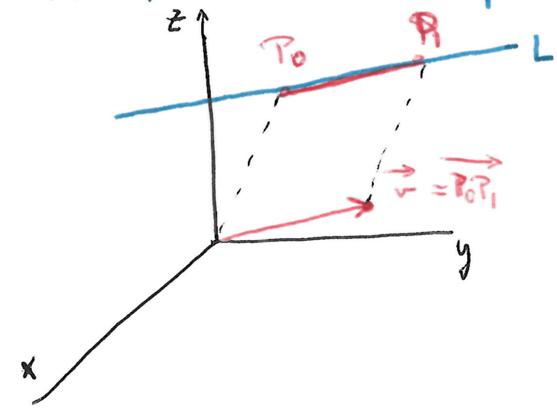


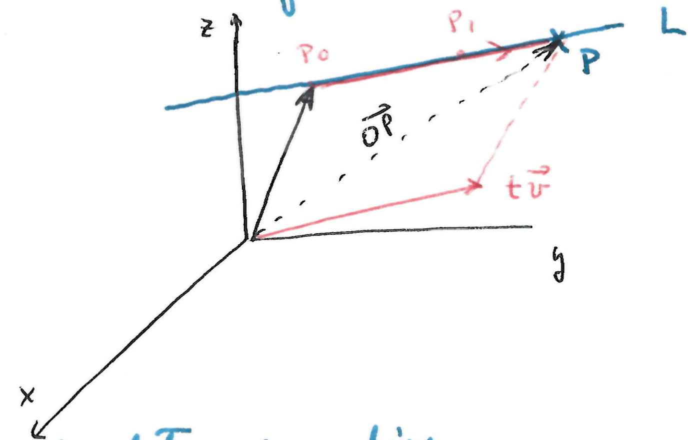
Vector form: direction of $L = \overrightarrow{P_0P_1}$. A point P belongs to L if and only if

$$\boxed{\overrightarrow{OP} = t\,\overrightarrow{P_0P_1} + \overrightarrow{OP_0} \quad \text{for some } t \text{ in } \mathbb{R}}$$

⟶ works for ANY slope!

• We use this idea to find equations describing lines in $\mathbb{R}^3$!





• 2 points P_0, P_1 in $\mathbb{R}^3$ uniquely determine a line

Vector Equation:

$$\boxed{\overrightarrow{OP} = \overrightarrow{OP_0} + t\,\overrightarrow{P_0P_1} \quad \text{for } t \text{ in } \mathbb{R}}$$

• Equivalently P belongs to L if and only if $\overrightarrow{PP_0}$ is parallel to $\overrightarrow{P_0P_1} = \vec{v}$

$$\boxed{\overrightarrow{P_0P} = t\,\overrightarrow{P_0P_1}\ (=t\vec{u}) \text{ for some } t \text{ in } \mathbb{R}}$$

where $\vec{u} = \overrightarrow{P_0P_1}$ is the direction of L

• P_0 is a (fixed) pt of L

• **Parametric equation:** Use the components of the vectors on each side:

Write $P_0 = (x_0, y_0, z_0)$ & $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

If $P = (x, y, z)$ then P lies in L if and only if $\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ for some t

Equivalently:

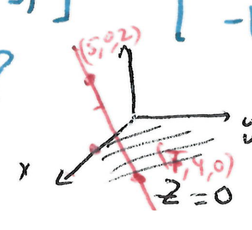
$$\begin{cases} x - x_0 = at \\ y - y_0 = bt \\ z - z_0 = ct \end{cases} \text{ for some } t \quad \longleftrightarrow \quad \begin{cases} x = \boxed{x_0} + \boxed{a}\,t \\ y = \boxed{y_0} + \boxed{b}\,t \\ z = \boxed{z_0} + \boxed{c}\,t \end{cases} \text{ for some } t$$

P_0 pt on L — direction of L

Example: Find the equation of the line which is parallel to $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ & passes through $(5, 0, 2)$.

Answer: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overrightarrow{OP} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} t+5 \\ 2t \\ -t+2 \end{bmatrix}$ $\quad P = (x, y, z)$

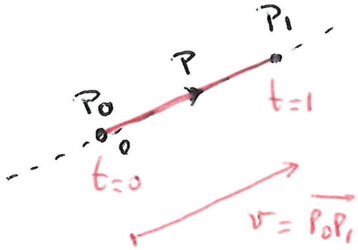
Q: Does this line intersect the xy-plane? If so, find the intersection pt.



$0 = z = -t + 2$ so $\boxed{t = 2}$

$P = (2+5, 2\cdot 2, 0) = (7, 4, 0)$

§2 Equation of a line segment



We restrict the equation of the line through P_0 & P_1 to specific values of t

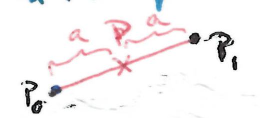
$$\boxed{\overrightarrow{P_0P} = t\,\overrightarrow{P_0P_1} \quad \text{for some } 0 \leq t \leq 1}$$

Why? • $P = P_0$ for $t=0$ ($\vec{0} = \overrightarrow{P_0P} = 0 \cdot \overrightarrow{P_0P_1}$)

• $P = P_1$ for $t = 1$ ($\overrightarrow{P_0P_1} = 1\,\overrightarrow{P_0P_1}$)

so middle values of t give the points in the segment

• Particular case: $\boxed{t = \frac{1}{2}}$ gives the midpoint between P_0 & P_1



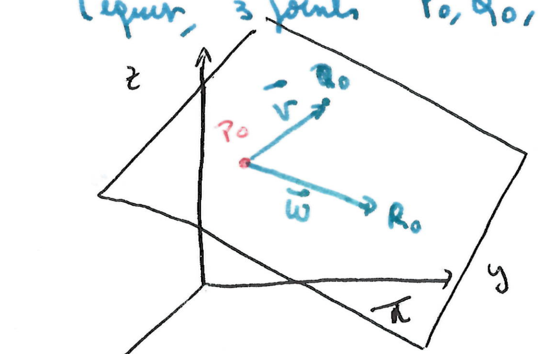
(defined as P for which $\dfrac{\|\overrightarrow{P_0P_1}\|}{2} = \|\overrightarrow{P_1P}\| = \|\overrightarrow{P_0P}\|$.

§3 Planes in 3-Space:

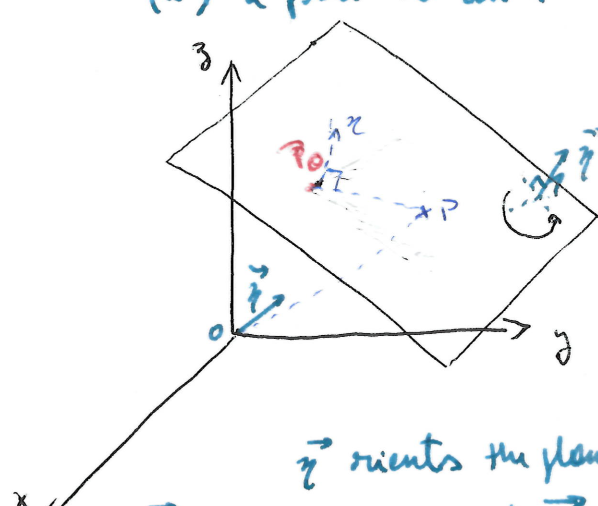
• We can determine a plane Π in $\mathbb{R}^3$ in 2 ways:

(i) a point P_0 & 2 nonparallel directions $(\vec{v}, \vec{w})$ (equiv, 3 points P_0, Q_0, R_0)

(ii) a point P_0 an a normal $\vec{\eta}$.



$$\begin{cases} \vec{v} = \overrightarrow{P_0Q_0} \\ \vec{w} = \overrightarrow{P_0R_0} \end{cases}$$



$\vec{\eta}$ orients the plane Π

$\vec{\eta}$ is normal to Π if $\vec{\eta}$ is perpendicular to every vector $\vec{u}$ in Π.

In particular $\vec{\eta} \perp \vec{v}$, $\vec{\eta} \perp \vec{w}$ so we can take $\boxed{\vec{\eta} = \vec{v} \times \vec{w}}$

(in general, if $\vec{v} \not\parallel \vec{w}$, we have $\vec{\eta} \parallel \vec{v} \times \vec{w}$)

We know $\vec{\eta} \cdot \vec{v} = \vec{\eta} \cdot \vec{w} = 0$. If $P_0 = (x_0, y_0, z_0)$, $\vec{\eta} = (a, b, c)$

Our vector equation for Π is $\boxed{\overrightarrow{P_0P} \cdot \vec{\eta} = 0}$ ($\overrightarrow{P_0P}$ is a vector in Π)

Explicitly, if $P = (x, y, z)$ (scalar form) $\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$ $\vec{\eta} = (a,b,c)$, $P_0 = (x_0, y_0, z_0)$