Lecture XIII : §2.4 (cont.) Planes, §3.1-2: Intro & Vector Space Prop. of $\mathbb{R}^n$

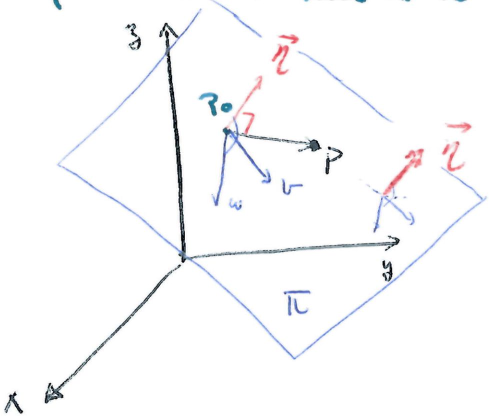
Recall: 2 descriptions of a plane Π in $\mathbb{R}^3$ → pt P_0 + 2 linearly independent directions $\vec{v}, \vec{w}$ (vectors in $\mathbb{R}^3$)

→ pt P_0 + normal direction $\vec{n}$

Can take $\vec{n} = \vec{v} \times \vec{w}$ (it is $\perp$ to any vector $\vec{u}$ in Π)

Note: If $\vec{n}$ is a normal, so is $-\vec{n}$

Vector equation: P lies in Π if and only if $\boxed{\overrightarrow{P_0P} \cdot \vec{n} = 0}$ (*)



Write $P_0 = (x_0, y_0, z_0)$, $P = (x, y, z)$

$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

(*) reads $\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$

Conversely: a linear equation in $\mathbb{R}^3$ determines a unique plane.

$\vec{n}$ = coefficients multiplying x, y & z in the linear eqn

Example Find the equation of the plane passing through $P_0 = (1,0,0)$, $R_0 = (1,1,1)$ & $Q_0 = (2,1,-1)$. Compute its intersection with the 3 coordinate planes.

Soln: $v = \overrightarrow{P_0Q_0} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$w = \overrightarrow{P_0R_0} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

so $\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} =$

$= \vec{i}\,(1-(-1)) - \vec{j}\,1 + \vec{k} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

so equation: $2(x-1) + (-1)(y-0) + (z-0) = 0$

$$\boxed{2x - y + z = 2}$$

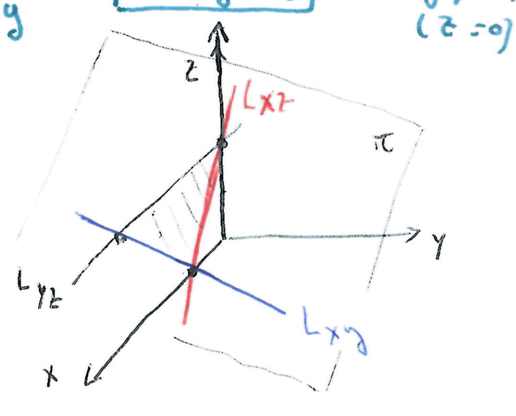
Check: P_0, Q_0 & R_0 satisfy the equation (eg P_0: $2\cdot 1 - 0 + 0 = 2$ ✓)

- The 3 intersections will be lines in the corresponding coordinate planes.

- xy-plane $\cap\ \Pi$: $\begin{cases} z = 0 & \text{(xy-plane)} \\ 2x - y + z = 2 \end{cases}$ so line $L_{xy} = \boxed{2x - y = 2}$ on xy-plane ($z=0$)

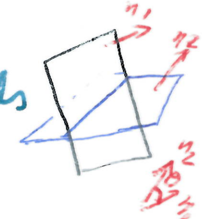
- yz-plane $\cap\ \Pi$: $\boxed{z = 2 + y}$ on yz-plane ($x = 0$)

- xz-plane $\cap\ \Pi$: $\boxed{2x + z = 2}$ on xz-plane ($y = 0$)



3 intersection pts between the 3 lines: $(0,0,2)$, $(0,-2,0)$, $(1,0,0)$

§1 Parallel & orthogonal planes



Def: Angle between 2 planes = (acute) angle between their normals

(use $|\vec{n}_1 \cdot \vec{n}_2| = \|\vec{n}_1\| \, \|\vec{n}_2\| \, |\cos\theta|$ to find θ with $0 \le \theta \le \frac{\pi}{2}$)

In particular: ① parallel planes = normals are parallel (that is, proportional)

② orthogonal planes = $\vec{n}_1 \perp \vec{n}_2$, so $\vec{n}_1 \cdot \vec{n}_2 = 0$.

Example: Find the parallel plane to $3x - 2y + 5z = 4$ passing through $(1,-1,1)$

Soln: normal $= \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$ $\rightsquigarrow$ $\pi : 3(x-1) - 2(y-(-1)) + 5(z-1) = 0$

$$\boxed{3x - 2y + 5z = 10}$$

Example: Find a plane orthogonal to $3x - 2y + 5z = 4$ passing through $\overset{P_0}{(1,0,0)}$ & $\overset{Q_0}{(1,1,0)}$

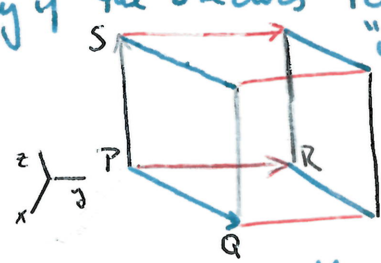
Soln: normal $\perp \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$,
& " $\perp \overrightarrow{P_0Q_0} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 3 & -2 & 5 \\ 0 & 1 & 0 \end{vmatrix} = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix} \rightsquigarrow \pi : -5(x-1) + 3(z-0) = 0$$

$$\boxed{-5x + 3z = -5} \text{ in } \mathbb{R}^3$$

§2 Coplanar points:

4 points P, Q, R, S are coplanar if and only if the 3 vectors $\overrightarrow{PQ}, \overrightarrow{PR}, \overrightarrow{PS}$ ($= \vec{u}, \vec{v}, \vec{w}$) form a flat parallelepiped, ie one of volume $= 0$



$$\boxed{\text{Formula for volume: } |\vec{u} \cdot (\vec{v} \times \vec{w})|}$$

Example Show that $(1,3,2)$, $(3,-1,6)$, $(5,2,0)$, $(3,6,-4)$ are coplanar

Soln 1: $\vec{u} = \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$ $\rightsquigarrow$ Vol $= \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix} \cdot \underbrace{\begin{vmatrix} i & j & k \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix}}_{= \begin{bmatrix} 12 \\ 20 \\ 14 \end{bmatrix}} = 0$ ✓

Soln 2: Find the equation of the plane through the 1st 3 pts & check the 4th one satisfies the equation

§3 The vector space $\mathbb{R}^n$ & its properties:

- Solutions to homogeneous systems of equations in $\mathbb{R}^n$
- vectors in $\mathbb{R}^2$ & $\mathbb{R}^3$

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- $\vec{0}$ lies in the space / is a solution
- add vectors / solutions & we remain a vector / solution of the same system.
- same for scalar multiplication

Note: Does not work for non-homogeneous systems!

These properties (plus some more) will characterize vector spaces ($\mathbb{R}^n$ will be our favorite example)

Theorem: Let $\mathbb{V} = \mathbb{R}^n$. For x, y, z vectors in $\mathbb{V}$ & α, β scalars, the following properties hold:

- **Closure properties:**
  - (C1) x, y in $\mathbb{V}$, then $x+y$ in $\mathbb{V}$
  - (C2) x in $\mathbb{V}$, α in $\mathbb{R}$, then αx in $\mathbb{V}$.
- **Addition properties:**
  - (A1) $x + y = y + x$ [Commutative]
  - (A2) $x + (y+z) = (x+y) + z$ [Associative]
  - (A3) $\vec{0}$ in $\mathbb{V}$ satisfies $x + \mathbb{0} = \mathbb{0} + x = x$ for all x in $\mathbb{V}$ [Neutral $= \mathbb{0}$]
  - (A4) given x in $\mathbb{V}$, there is an element $(-x)$ in $\mathbb{V}$ satisfying $x + (-x) = \mathbb{0}$. $(-x = (-1)x)$
- **Scalar Mult properties:**
  - (M1) $\alpha(\beta x) = (\alpha\beta)x$ [Assoc]
  - (M2) $\alpha(x+y) = \alpha x + \alpha y$ [Distributive]
  - (M3) $(\alpha+\beta)x = \alpha x + \beta x$ [———]
  - (M4) $1x = x$ for all x in $\mathbb{V}$.

Def A subset $\mathbb{V}$ of $\mathbb{R}^n$ satisfying these properties is called a subspace of $\mathbb{R}^n$.

Note: A1, A2, M1 — M4 are inherited from $\mathbb{R}^n$, so only need to check C1, C2, A3, [A4] (follows from C2). So

Thm 2: A subset $\mathbb{V}$ of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if and only if

(S1) The zero vector $\vec{0}$ is in $\mathbb{V}$

(S2) $x+y$ lies in $\mathbb{V}$ whenever x, y are in $\mathbb{V}$

(S3) αx ——— x is in $\mathbb{V}$ & α is any scalar in $\mathbb{R}$.

META **Example:** A solution set to a homogeneous system of m equs in $\mathbb{R}^n$. $(A\,\underline{x} = \mathbb{0})$

Why? (S1) The trivial soln is $\mathbb{0}$ in $\mathbb{R}^n$ ✓

(S2) $\left.\begin{array}{l} Ax = \mathbb{0} \text{ in } \mathbb{R}^m \\ Ay = \mathbb{0} \text{ in } \mathbb{R}^m \end{array}\right\}$ so $A(x+y) = \underbrace{Ax}_{=\mathbb{0}} + \underbrace{Ay}_{=\mathbb{0}} = \mathbb{0}$ in $\mathbb{R}^m$ so $\underline{x+y}$ is a soln!

(S3) $A(\alpha x) \underset{\text{alg prop}}{=} \alpha(Ax) = \alpha \cdot \mathbb{0} = \mathbb{0}$ in $\mathbb{R}^m$ so $\alpha \underline{x}$ is a soln!

Example 1: A line L in $\mathbb{R}^3$ through $(0,0,0)$, eg $\begin{cases} x + y + z = 0 \\ x + 2y - z = 0 \end{cases}$

(S1) $(0,0,0)$ in the line ✓

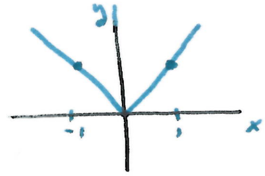
(S2) $[x_1, y_1, z_1]^t$ in $L \Rightarrow$ $(x_1+x_2) + (y_1+y_2) + (z_1+z_2) = (x_1+y_1+z_1) + (x_2+y_2+z_2) = 0+0 = 0$

$[x_2, y_2, z_2]^t$ in L $\quad (x_1+x_2) + 2(y_1+y_2) - (z_1+z_2) = (x_1+2y_1-z_1) + (x_2+2y_2-z_2) = 0+0 = 0$

So $[x_1+x_2,\ y_1+y_2,\ z_1+z_2]^t$ in L.

(S3) α in $\mathbb{R}$: $\begin{cases} \alpha x_1 + \alpha y_1 + \alpha z_1 = \alpha(x_1+y_1+z_1) = \alpha \cdot 0 = 0 \\ \alpha x_1 + 2\alpha y_1 - \alpha z_1 = \alpha(x_1+2y_1-z_1) = \alpha \cdot 0 = 0 \end{cases}$

• <u>Non example</u>: Graph of $|x|$ in $\mathbb{R}$



(S1) holds

(S2) $\underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\underline{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\leadsto$ so $\underline{x} + \underline{y} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ not in the graph of the function!

• <u>Non example</u>: $\mathbb{W} = \left\{ \underline{x} : \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ 3 \end{bmatrix} \quad x_1, x_2 \text{ in } \mathbb{R} \right\}$ = the plane in $\mathbb{R}^3$ with equation $z = 3$.

(S1) does not hold.

• <u>Non example</u>: $\mathbb{W} = \left\{ \underline{x} : x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad x_1, x_2 \text{ are integers} \right\}$

(S1) holds

(S2) holds (integers are closed under sums)

(S3) does <u>not</u> hold : $\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ is <u>not</u> in $\mathbb{W}$ but $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is.

<u>Next Time</u>: Examples coming from • linear combinations of vectors in $\mathbb{R}^n$
• matrices.