# Lecture XIV: §3.3 Examples of subspaces of $\mathbb{R}^n$

**Recall**: A subset $\mathbb{W}$ of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if and only if:

(S1) The zero vector $\vec{0}$ is in $\mathbb{W}$

(S2) $\vec{x}+\vec{y}$ lies in $\mathbb{W}$ whenever $\vec{x}, \vec{y}$ are in $\mathbb{W}$

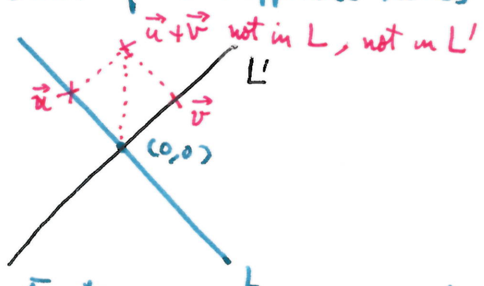
(S3) $\alpha\vec{x}$ ——— $\vec{x}$ is in $\mathbb{W}$ & $\alpha$ is any scalar in $\mathbb{R}$

**Example**: A line in $\mathbb{R}^3$ through the origin ( direction $= \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$, defined by $\begin{cases} x+y+z=0 \\ x+2y-z=0 \end{cases}$)

**Non example**: graph of $f(x)=|x|$.

**Non-example**: Union of 2 different lines in $\mathbb{R}^2$ through the origin

Why?




**TODAY**: Important examples coming from matrices, spans of vectors.

## §1 The Span of a Subset

**Def**: Given vectors $\vec{v}_1, \ldots, \vec{v}_r$ in $\mathbb{R}^n$, a vector $\vec{y}$ in $\mathbb{R}^n$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_r$ if we can find scalars $\alpha_1, \ldots, \alpha_r$ in $\mathbb{R}$ giving

$$\vec{y} = \alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r .$$

We write $\mathbb{W} = Sp(\vec{v}_1, \ldots, \vec{v}_r)$ = set of all linear comb. of $\vec{v}_1 \cdots \vec{v}_r$.

**Thm 1**: The set $\mathbb{W} = Sp(\vec{v}_1, \ldots, \vec{v}_r)$ is a subspace of $\mathbb{R}^n$

**Proof**: Need to show (S1), (S2) (S3):

(S1): $\vec{0} = 0\vec{v}_1 + \cdots + 0\vec{v}_r$ ✓

(S2): $\vec{x}$ in $\mathbb{W}$: $\vec{x} = \alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r$ — $\alpha_1, \ldots, \alpha_r$ scalars

$\vec{y}$ in $\mathbb{W}$: $\vec{y} = \beta_1\vec{v}_1 + \cdots + \beta_r\vec{v}_r$ — $\beta_1, \ldots, \beta_r$ ———

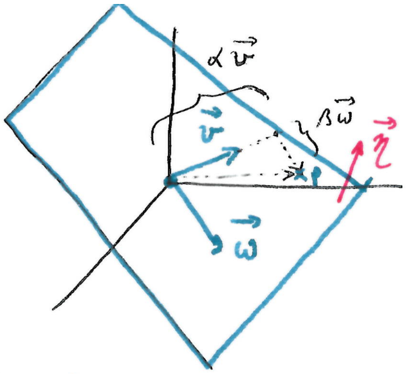
$\vec{x}+\vec{y} = (\alpha_1+\beta_1)\vec{v}_1 + \cdots + (\alpha_r+\beta_r)\vec{v}_r$ — $(\alpha_1+\beta_1), \ldots, (\alpha_r+\beta_r)$ scalars

(S3) $\vec{x}$ in $\mathbb{W}$ (as above) $\rightsquigarrow$ $\alpha\vec{x} = (\alpha\alpha_1)\vec{v}_1 + \cdots + (\alpha\alpha_r)\vec{v}_r$ — $\alpha\alpha_1, \ldots, \alpha\alpha_r$ scalars. □

**Example 1** A line L through the origin in $\mathbb{R}^n$: vector eqn: $\vec{x} = \underset{\text{point on } L}{\vec{0}} + t\vec{v}$ for $\vec{v} = \text{dir}(L)$, $t$ parameter

so $L = Sp(\vec{v})$

**Example 2**: A plane $\pi$ in $\mathbb{R}^3$ passing through the origin.



Eqn $\overrightarrow{OP} \cdot \vec{\eta} = 0$

$\vec{v}, \vec{w}$ noncollinear vectors, so $P$ lies in $\pi$ if and only if $\overrightarrow{OP} = \alpha \vec{v} + \beta \vec{w}$ for some $\alpha, \beta$ in $\mathbb{R}$.

Conclusion: plane $= Sp(\vec{v}, \vec{w})$.

Why? (via example) $\vec{v} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\rightsquigarrow$ $\vec{\eta} = \vec{v} \times \vec{w} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$ $\rightsquigarrow$ Eqn: $-y_1 + 2y_2 - y_3 = 0$

$y$ in $Sp(\vec{v}, \vec{w})$, so $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = a \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2a + b \\ -a + b \\ b \end{bmatrix}$

Given $\vec{y}$ we must find $a, b$ that solves the system of 3 eqns in 2 unknowns

$\Rightarrow$ Augmented matrix

$$\left[\begin{array}{cc|c} -2 & 1 & y_1 \\ -1 & 1 & y_2 \\ 0 & 1 & y_3 \end{array}\right] \xrightarrow[R_1 \to -R_1]{R_2 \leftrightarrow R_1} \left[\begin{array}{cc|c} 1 & -1 & -y_2 \\ -2 & 1 & y_1 \\ 0 & 1 & y_3 \end{array}\right] \xrightarrow[R_2 \to R_2 + 2R_1]{} \left[\begin{array}{cc|c} 1 & -1 & -y_2 \\ 0 & -1 & y_1 - 2y_2 \\ 0 & 1 & y_3 \end{array}\right] \xrightarrow[R_2 \to -R_2]{R_3 \to R_3 + R_2} \left[\begin{array}{cc|c} 1 & -1 & -y_2 \\ 0 & 1 & -y_1 + 2y_2 \\ 0 & 0 & \boxed{y_1 - 2y_2 + y_3} \end{array}\right]$$

The system is consistent if and only if $\boxed{y_1 - 2y_2 + y_3 = 0} = (-\text{Eqn})$ ✓

## §2. The Null Space of a Matrix:

**Def** Given an $m \times n$ matrix $A$, the solution set of $Ax = \mathbb{0}$ in $\mathbb{R}^n$ is the Null Space (or kernel) of $A$.

$$\mathcal{N}(A) = \{ x : Ax = 0, x \text{ in } \mathbb{R}^n \}$$

**Thm 2:** $\mathcal{N}(A)$ is a (vector) subspace of $\mathbb{R}^n$

**Proof:** (S1) $\mathbb{0}$ = trivial solution is a solution ✓

(S2) given 2 solutions $\vec{u}, \vec{v}$, then $A(\vec{u} + \vec{v}) = \underbrace{A\vec{u}}_{=\vec{0}} + \underbrace{A\vec{v}}_{=\vec{0}} = \vec{0}$ ✓ so $(\vec{u} + \vec{v})$ is a soln!

(S3) given a solution $\vec{u}$, $\alpha$ scalar then $A(\alpha \vec{u}) = \alpha(A\vec{u}) = \alpha \cdot \vec{0} = \vec{0}$ ✓ so $\alpha\vec{u}$ is a soln!

**Note:** The general form of a solution to $Ax = \vec{0}$ is ~~the~~ the ~~form~~ $\vec{x} = x_{i_1} \vec{v}_1 + \cdots + x_{i_r} \vec{v}_r$ where $x_{i_1}, \ldots, x_{i_r}$ are the independent variables so the Null Space $= Sp(\vec{v}_1, \ldots, \vec{v}_r)$

**Example:** $A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 3 & 2 & 8 & 5 \\ -1 & -2 & -4 & +1 \end{bmatrix} \rightsquigarrow [A | 0] \underset{\text{row equiv}}{\sim} \begin{array}{c} \begin{array}{cccc} & & \text{indep} & \\ & & \downarrow & \downarrow \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$

$x_1 = -2x_3 - 3x_4$
$x_2 = x_3 + 2x_4$ $\rightsquigarrow$ $\vec{x} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ $\rightsquigarrow$ $\mathcal{N}(A) = Sp\left( \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right)$

Example: A plane in $\mathbb{R}^3$ through $(0,0,0)$ has eqn $ax+by+cz=0$, $\vec{n}=\begin{bmatrix} a\\ b\\ c\end{bmatrix}$

Claim: The plane $= N([a\ b\ c])$. where some $a,b,c \neq 0$

## §3. The Range of a matrix

Def: Given an $(m\times n)$ matrix $A$, the range of $A$ is the set of vectors:

$$\mathcal{R}(A) = \{ y \text{ in } \mathbb{R}^m : y = Ax \quad \text{for some } x \text{ in } \mathbb{R}^n\}$$

Note: write $A = [\vec{v}_1 \cdots \vec{v}_n]$ for $\vec{v}_1, \dots, \vec{v}_m = \text{col}(A)$: vectors in $\mathbb{R}^m$, then

$$Ax = x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n \quad \& \ x_1, \dots, x_n \text{ are arbitrary scalars}$$

We conclude: $\boxed{\mathcal{R}(A) = Sp(\text{columns of } A)}$ so we call it the Column Space of A

Thm 3: The range of $\mathcal{R}(A)$ is a subspace of $\mathbb{R}^n$.

Example: Describe the range of $A = \begin{bmatrix} 1 & -1 & 1 & 1\\ 2 & -1 & 4 & 0\\ 1 & 1 & 5 & -3\end{bmatrix}$

Equiv: $\vec{y} = A\vec{x}$ $\rightsquigarrow [A|y] = \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & y_1\\ 2 & -1 & 4 & 0 & y_2\\ 1 & 1 & 5 & -3 & y_3\end{array}\right] \xrightarrow[R_3\to R_3-R_1]{R_2\to R_2-2R_1} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & y_1\\ 0 & 1 & 2 & -2 & y_2-2y_1\\ 0 & 2 & 4 & -4 & y_3-y_1\end{array}\right]$

$\xrightarrow[R_3\to R_3-2R_2]{} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & y_1\\ 0 & 1 & 2 & -2 & y_2-2y_1\\ 0 & 0 & 0 & 0 & 3y_1-2y_2+y_3\end{array}\right]$

The system is compatible if and only if $\boxed{3y_1-2y_2+y_3=0}$ = plane in $\mathbb{R}^3$

so $y_3 = -3y_1+2y_2$ $\quad y = \begin{bmatrix} y_1\\ y_2\\ -3y_1+2y_2\end{bmatrix} = y_1\begin{bmatrix}1\\0\\-3\end{bmatrix} + y_2\begin{bmatrix}0\\1\\2\end{bmatrix}$ in $Sp\left(\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right)$

## §4. The Row Space of a matrix

Def: Given $A$ $(m\times n)$ matrix, we define the Row Space of A as the span of its $m$ rows (vectors in $\mathbb{R}^n$, transposed)

Eg $A = \begin{bmatrix} 1 & 2 & 4\\ 5 & 7 & 8\end{bmatrix}$ $\quad \mathcal{R}(A) = Sp\left(\begin{bmatrix}1\\5\end{bmatrix}, \begin{bmatrix}2\\7\end{bmatrix}, \begin{bmatrix}4\\8\end{bmatrix}\right)$

$\quad Row(A) = Sp\left(\begin{bmatrix}1\\2\\4\end{bmatrix}, \begin{bmatrix}5\\7\\8\end{bmatrix}\right)$

Q: What happens under elementary Row operations? Answ: Row(A) is preserved!

Thm 4: If $A \underset{\text{row equiv}}{\sim} B$, then $A$ & $B$ have the same row space.

Proof (idea): Show that this property holds if A & B are related by single elementary row operation.

(E1) Swapping rows clearly preserves the row space

(E2) Multiplying a row (say $R_1$) by a scalar $\alpha \neq 0$:

$$Sp(\alpha R_1, R_2, \ldots, R_m) = Sp(R_1, R_2, \ldots, R_m)$$

($\subseteq$) because $x = \alpha_1(\alpha R_1) + \alpha_2 R_2 + \cdots + \alpha_m R_m = (\alpha_1 \alpha) R_1 + \alpha_2 R_2 + \cdots + \alpha_m R_m$.

($\supseteq$) " $x = \alpha_1 R_1 + \cdots + \alpha_m R_m = \frac{\alpha_1}{\alpha}(\alpha R_1) + \cdots + \alpha_m R_m$

(E3) Multiplying a row (say $R_1$) & adding the result to a row (say $R_2$)

$$Sp(R_1, \alpha R_1 + R_2, \ldots, R_m) = Sp(R_1, R_2, \ldots, R_m)$$

($\subseteq$) because $x = \alpha_1 R_1 + \alpha_2(\alpha R_1 + R_2) + \cdots + \alpha_m R_m = (\alpha_1 + \alpha \alpha_2) R_1 + \alpha_2 R_2 + \cdots + \alpha_m R_m$

($\supseteq$) " $x = \alpha_1 R_1 + \alpha_2 R_2 + \cdots + \alpha_m R_m = (\alpha_1 - \alpha \alpha_2) R_1 + \alpha_2(\alpha R_1 + R_2) + \alpha_3 R_3 + \cdots + \alpha_m R_m$

__Observation__ We can use row operations to find a better set of generators of Row(A), $\square$ and in general for $Sp(v_1, \ldots, v_m) = \mathbb{W}$ in $\mathbb{R}^n$

__Example:__ $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$, $v_4 = \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}$ (4 generators)

- Write $A$ with rows $v_1^t, v_2^t, v_3^t, \ldots, v_m^t$. ($A$ ($m \times n$) matrix)
- Find $A \sim B$ with $B$ in reduced echelon form.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \\ -1 & -1 & -4 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 + R_1}]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 0 & -1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_2 \\ R_4 \to R_4 + R_2 \\ R_2 \to -R_2}]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & +1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 \to R_1 - 2R_2]{} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Take the nonzero rows of $B$, transpose them & we get a better set of generators for $\mathbb{W}$

New set $= \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ +1 \\ -3 \end{bmatrix}$. $\quad \mathbb{W} = Sp\left(\begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}\right)$. (2 generators)

__Next Time__: See what "better set" means (Basis for the subspace $\mathbb{W}$).