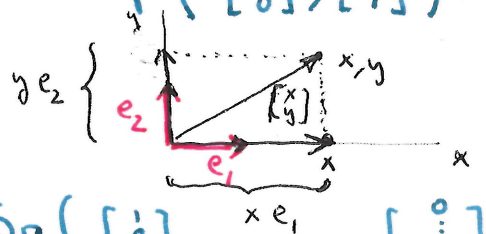
Lecture XV §3.5 Bases for Subspaces

Recall: In $\mathbb{R}^2 = Sp\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) := \left\{ \begin{bmatrix}x\\y\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} \text{ for some } a,b \text{ in } \mathbb{R}\right\}$

(in this case $a = x$, $b = y$)



Similarly:

$$\mathbb{R}^n = Sp\Big(\underbrace{\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}}_{e_1}, \ldots\ldots, \underbrace{\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}}_{e_n}\Big) = Sp(e_1, \ldots, e_n)$$ because $\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} = x_1e_1 + x_2e_2 + \cdots + \cdots + x_ne_n$

and $x_1, \ldots, x_n$ are the <u>unique</u> scalars that give $\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}$ as a linear combination of $e_1, \ldots, e_n$

Equivalent, if we remove any e_i from $\{e_1, \ldots, e_n\}$, the subspace spans a <u>proper</u> subspace of $\mathbb{R}^n$. $\boxed{\{e_1, \ldots, e_n\} \text{ is a basis for } \mathbb{R}^n}$

A <u>basis</u> for $\mathbb{R}^n$ will be a minimal generating set.
(or for a subspace W)

<u>Last time</u> $W = Sp\left(\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\3\\5\end{bmatrix}, \begin{bmatrix}3\\5\\6\end{bmatrix}, \begin{bmatrix}-1\\-1\\-4\end{bmatrix}\right) = Sp\Big(\overbrace{\begin{bmatrix}1\\0\\7\end{bmatrix}, \begin{bmatrix}0\\1\\-3\end{bmatrix}}^{l.i.}\Big)$

Why: $W = \text{Row}\begin{pmatrix}1&2&1\\2&3&5\\3&5&6\\-1&-1&-4\end{pmatrix}$ & $\begin{pmatrix}1&2&1\\2&3&5\\3&5&6\\-1&-1&-4\end{pmatrix} \underset{\text{row equiv}}{\sim} \begin{pmatrix}1&0&7\\0&1&-3\\0&0&0\\0&0&0\end{pmatrix}$

& Row operations preserve the row space.

<u>Claim</u>: $\left\{\begin{bmatrix}1\\0\\7\end{bmatrix}, \begin{bmatrix}0\\1\\-3\end{bmatrix}\right\}$ is a basis for W.

§1 Spanning Sets:

Def: Let W be a <u>subspace</u> of $\mathbb{R}^n$, $S = \{w_1 \cdots w_r\}$ a subset of W. The set S is a <u>spanning set</u> for W (or S spans W) if every vector w in W is a linear combination of $\{w_1, \ldots, w_r\}$. In short: $W = Sp(w_1, \ldots, w_r)$

<u>Example 1</u>: Determine if $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\0\\-7\end{bmatrix}, \begin{bmatrix}2\\7\\0\end{bmatrix}\right\}$ spans $\mathbb{R}^3$.

<u>Soln</u>: Want to write <u>any</u> $v = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}-1\\0\\-7\end{bmatrix} + c\begin{bmatrix}2\\7\\0\end{bmatrix} = \begin{bmatrix}1&-1&2\\2&0&7\\3&-7&0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}$

So $\left[\begin{array}{ccc|c}1&-1&2&v_1\\2&0&7&v_2\\3&-7&0&v_3\end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c}1&-1&2&v_1\\0&2&3&v_2-2v_1\\0&-4&-6&v_3-3v_1\end{array}\right] \xrightarrow[R_2 \to R_2/2]{R_3 \to R_3 + 2R_2} \left[\begin{array}{ccc|c}1&-1&2&v_1\\0&1&3/2&\frac{v_2-2v_1}{2}\\0&0&0&v_3+2v_2-7v_1\end{array}\right]$

give a consistent system

$0 = v_3 + 2v_2 - 7v_1$

consistent if and only if $\boxed{v_3 + 2v_2 - 7v_1 = 0}$

So a $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ lies in the span if and only if $v_3 + 2v_2 - 7v_1 = 0$

In particular $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ does not! $(1+2-7 = -4 \neq 0)$ so the vectors don't span $\mathbb{R}^3$

They span the plane in $\mathbb{R}^3$ with eqn. $v_3 + 2v_2 - 7v_1 = 0$.

<u>Example 2</u>: Determine if $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -7 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 1 \end{bmatrix} \right\}$ spans $\mathbb{R}^3$.

<u>Soln</u>: $\left[\begin{array}{ccc|c} 1 & -1 & 2 & v_1 \\ 2 & 0 & 8 & v_2 \\ 3 & -7 & 1 & v_3 \end{array}\right] \underset{\text{row equiv}}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 2 & v_1 \\ 0 & 1 & 3/2 & \frac{v_2 - 2v_1}{2} \\ 0 & 0 & 1 & v_3 + 2v_2 - 7v_1 \end{array}\right] \underset{\text{row}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & v_3 + 2v_2 - 7v_1 \end{array}\right]$

↳ no conditions! so Answer = YES!

<u>Q</u>: Spanning set of $Null(A) = \{ x : Ax = 0 \}$?

Solve the system $A \underset{\text{row}}{\sim} B$ in Red Echelon Form, so spanning set comes from the independent variables

<u>Eg</u>: $A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 1 & 5 & 4 \\ 1 & 2 & 4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 - 3x_4 \\ -x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

indep ↳ general soln

So $Null(A) = Sp\left(\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right)$

<u>Obs</u>: Spanning set of $\mathcal{R}(A) = \{ y : y = Ax \text{ for some } x \text{ in } \mathbb{R}^n \} = Sp(col_1 A, \ldots, col_n A)$

because $Ax = x_1 \, col_1 A + \cdots + x_n \, col_n A$.

<u>Obs</u>: Spanning set of $Row(A)$ = rows of A.

<u>§2 Minimal Spanning Sets</u>:

<u>Example 1</u>: $W = \{ x_3 + 2x_2 - 7x_1 = 0 \} = Sp\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix} \right)$ 3 <u>l.d.</u> vectors

$$\begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix} = \frac{7}{2} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} -1 \\ 0 \\ -7 \end{bmatrix}$$

So $W = Sp\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -7 \end{bmatrix} \right)$

<u>Conclusion</u>: If $S = \{ w_1, \ldots, w_r \}$ are l.d. vectors, then they are not a minimal spanning set. By removing one at a time we can generate the same

subspace with l.i. vectors (remove w_i from S if

$w_i = \alpha_1 w_1 + \cdots + \alpha_{i-1} w_{i-1} + \alpha_{i+1} w_{i+1} + \cdots + \alpha_r w_r$ &

then repeat)

Algorithm: Start from $S = \{w_1, \ldots, w_r\}$

Step 1: Is S l.i? If YES, then S = minimal spanning set for S ✓

If NO, find a w_i that is a linear combination of the remaining $S' = \{w_1, \ldots, \underset{\text{removed}}{\widehat{w_i}}, \ldots, w_r\}$

Step 2: Replace S with S' and repeat step 1.

Output: A minimal spanning set.

Def: A basis B of a nonzero subspace $\mathbb{W}$ of $\mathbb{R}^n$ is a minimal spanning set.

Equivalently, B is (1) spanning set for $\mathbb{W}$

(2) linearly independent.

Obs: If $\mathbb{W} = \{\underline{0}\}$, then $B = \{\underline{0}\}$ but $\{\underline{0}\}$ is l.d, so the zero subspace has no basis.

Examples: (1) $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$ (call it the canonical basis for $\mathbb{R}^n$)

(2) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\0\\-7\end{bmatrix}\right\}$ is a basis of $\{x_3 + 2x_2 - 7x_1 = 0\}$ = plane in $\mathbb{R}^3$

(3) $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^3$

Thm [Uniqueness of Representation]

Given a basis $B = \{v_1, \ldots, v_d\}$ of $\mathbb{W}$ (nonzero subspace of $\mathbb{R}^n$), if v is in $\mathbb{W}$, then v can be represented uniquely in terms of B. That is there are unique scalars $\alpha_1, \ldots, \alpha_d$ such that $v = \alpha_1 v_1 + \cdots + \alpha_d v_d$

We call these scalars the coordinates of v with respect to the basis B

Eg: $\underline{x} = x_1 \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} + \cdots + x_n \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$ usual coords = coords with respect to the canonical basis of $\mathbb{R}^n$.

• **Why?** If 2 representations exist for v, then

$v = \alpha_1 v_1 + \cdots + \alpha_n v_n = \beta_1 v_1 + \cdots + \beta_n v_n$, take their difference & set $\underline{0} = (\alpha_1 - \beta_1) v_1 + \cdots + (\alpha_n - \beta_n) v_n$ nontrivial combination, so B is l.d, this can't happen!