Lecture XVI: §3.4 (cont.) Bases for Subspaces, §3.5 Dimension

Recall: Given a nonzero subspace W of $\mathbb{R}^n$ a set $B = \{v_1, \dots, v_d\}$ is a basis for W if (i) B spans W (every v in W is a linear combination of $v_1, \dots, v_d$).
(ii) B is l.i.

Thm: Every v in W has a unique representation in terms of $\underset{\text{(basis)}}{B}$, that is

$$v = \alpha_1 v_1 + \dots + \alpha_d v_d$$

& $(\alpha_1, \dots, \alpha_d)$ are unique, We call them the coordinates of v with respect to the basis B.

Eg: For $\mathbb{R}^n$; $\{e_1, \dots, e_n\}$ is the (canonical) basis for $\mathbb{R}^n$ &

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + \dots + x_n e_n$$

so the usual coords are the coords w.r.t. the canonical basis.

§: How to find a basis from a spanning set $\{w_1, \dots, w_r\}$?

METHOD 1: Write $W = \underset{\hookrightarrow \text{range}}{R(A)}$ for $A = [w_1 \cdots w_r]$ of size $n \times r$.

Step 1: Find dependence relations among the r-vectors & remove one per dependence relation.
For this $A \underset{\text{row equiv}}{\sim} B$ with B in red. echelon form.

Step 2: Basis $= \{ w_i$'s indexed by DEPENDENT variables in Null(A). $\}$

Example 1: $V = \mathrm{Sp}\left(\underset{= v_1}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}, \underset{= v_2}{\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}}, \underset{= v_3}{\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}}, \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix} = v_4 \right)$ Dependencies = solns to $Ax = 0$

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 5 & -1 \\ 1 & 5 & 6 & -4 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1}]{} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 3 & 3 & -3 \end{bmatrix} \xrightarrow[\substack{R_1 \to -R_1 \\ R_3 \to R_3 - 3R_1}]{} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_1 \to R_1 - 2R_2]{} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

dependent

Solns to $Ax = 0$ are Soln to $Bx = 0$

$$\begin{cases} x_1 + x_3 + x_4 = 0 \\ x_2 + x_3 - x_4 = 0 \end{cases} \qquad \begin{aligned} x_1 &= -x_3 - x_4 \\ x_2 &= -x_3 + x_4 \end{aligned}$$

so general soln $\underline{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ for any x_3, x_4.

This means: $-v_1 - v_2 + v_3 = 0$ & $-v_1 + v_2 + v_4 = 0$ are the dependency relations

So we get $v_3 = v_1 + v_2$ $\rightsquigarrow$ $\mathbb{W} = Sp(v_1, v_2)$ & $\{v_1, v_2\}$ as l.i (have no reln!)

$v_4 = v_1 - v_2$

Basis $= \{v_1, v_2\}$.

Note: x_1, x_2 were the dependent variables!

METHOD 2: Write $\mathbb{W} = RowSp(A)$ for $A = \begin{bmatrix} w_1^T \\ \vdots \\ w_r^T \end{bmatrix}$ of size $n \times r$.

Step 1 Find $A \underset{\text{row equiv}}{\sim} B$ in (R)E.F.

Step 2 Basis of A = {nonzero rows of B}.

Notes:
- The set get has no relation to $\{w_1, \dots, w_r\}$.
- This works since $RowSp(A) = RowSp(B)$ (Thm from Lecture XIV)

Example (revisited): $\mathbb{W}$ as in Example 1

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \\ -1 & -1 & -4 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 + R_1}]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_2 \\ R_4 \to R_4 + R_2 \\ R_2 \to -R_2}]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Big\} \text{ nonzero rows}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right.$$

- Can take B to be in REF $B \sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right\}$

§2. Dimension:

In the examples above, we see that the bases constructed have the same size!

Def The dimension of a subspace $\mathbb{W}$ of $\mathbb{R}^n$ is the number of vectors in a basis B of $\mathbb{W}$.

Obs: If $\mathbb{W} = \{0\}$, then $\mathbb{W}$ has no basis because $\{0\}$ is l.d. So the dim $= 0$

Note: $\dim \mathbb{R}^n = n$ for any $n \geq 1$ ($\{e_1, \dots, e_n\}$ is a basis)

Q: Why $\overset{\text{do}}{\wedge}$ all basis of $\mathbb{W}$ have the same number of vectors?

Theorem 1: Let $\mathbb{W}$ be a nonzero subspace of $\mathbb{R}^n$ & $S = \{w_1, \dots, w_p\}$ a spanning set for $\mathbb{W}$. Then, any set with $p+1$ or more elements in $\mathbb{W}$ is linearly dependent.

Pf/ Pick $v_1, \dots, v_m$ in $\mathbb{W}$ with $m \geq p+1$ & write each one as a lin comb of S.

$$\left\{ \begin{array}{l} v_1 = a_{11} w_1 + \dots + a_{p1} w_1 \\ \vdots \\ v_m = a_{1m} w_1 + \dots + a_{pm} w_n \end{array} \right.$$

Want to find a nontrivial solution to $\alpha_1 v_1 + \cdots + \alpha_m v_m = 0$.

Rewrite it in terms of (a_{ij}) & $\{w_1, \dots, w_p\}$:

$$\text{Since} \quad \underset{\text{size } n\times m}{\begin{bmatrix} v_1 \cdots v_m \end{bmatrix}} = \underset{\text{size } n\times p}{\begin{bmatrix} w_1 \cdots w_p \end{bmatrix}} \overset{\text{size } p\times m}{\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{p1} & \cdots & a_{pm} \end{bmatrix}}$$

$$\text{Then} \quad \vec{0} \overset{?}{=} \begin{bmatrix} v_1 \cdots v_m \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} w_1 \cdots w_p \end{bmatrix} \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{p1} & \cdots & a_{pm} \end{bmatrix}}_{=A} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}$$

Lecture VII (Thm 1)

The columns of A are $m > p$ vectors in $\mathbb{R}^p$ so they are l.d. In particular A is singular & we can find $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}$ nonzero with $A\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = 0$ in $\mathbb{R}^p$. Then this $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}$ gives a nontrivial relation among $v_1 \cdots v_m$ so these vectors are l.d. □.

Corollary: If $\mathbb{W}$ has a basis B with p elements, any basis of $\mathbb{W}$ must have p elem.

Pf/ If we pick another basis B' since it's l.i it has $\leq p$ elements. If it had less, then B would be l.d. (it has more elements than B'). But this can't happen if B is a basis. So B' has also p elements. □.

• Lecture VII (Thm 1): A set of $m > p$ vectors in $\mathbb{R}^p$ is l.d.

Similar situation happens for a general subspace of $\dim = p$.

Thm 2: Given a nonzero subspace $\mathbb{W}$ of $\mathbb{R}^n$ of $\dim \mathbb{W} = p$, we have:

(1) A set of $p+1$ or more vectors in $\mathbb{W}$ is linearly dependent

(2) Any set of $p-1$ or fewer vectors in $\mathbb{W}$ can't span $\mathbb{W}$.

(3) Any set of p linearly independent vectors in $\mathbb{W}$ is a basis for $\mathbb{W}$

(4) Any set of p vectors in $\mathbb{W}$ that spans $\mathbb{W}$ is a basis for $\mathbb{W}$.

Application: See next page

Pf/ (1) Follows from Thm 1.

(2) If a set S like this spanned, then a basis with $p > \text{size}(S)$ elements would be l.d (by Thm 1) This can't happen!

(3) If it were not a spanning set, we could add a vector v to it and remain l.i, so we would have a set contradicting (1).

(4) If it weren't l.i, we could remove one or more vectors and still span $\mathbb{W}$. Therefore $\dim \mathbb{W} < p$, contradicting our original assumption.

§3. The Rank of a matrix:

A matrix of size $m \times n$ $\rightsquigarrow$ $\mathcal{N}ull(A)$ subspace of $\mathbb{R}^n$ dim =: nullity of A

$\rightsquigarrow$ $\underset{\text{range}}{\mathcal{R}}(A)$ subspace of $\mathbb{R}^m$ dim =: rank(A)

Q: How to compute nullity and rank?

- For Nullity : Find $A \underset{\text{row equiv}}{\sim} B$ R.E.F. then nullity = # independent variables

Example 1 (revisited) $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 5 & -1 \\ 1 & 5 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \overset{\downarrow}{1} & \overset{\downarrow}{1} \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ 2 indep. variables nullity = 2.

- For Range : Find $A^T \underset{\text{row equiv}}{\sim} B$ REF then rank = # nonzero rows of A.

Example 1 (revisited) $A^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \\ -1 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ rank = 2.

Thm 3 The rank of A = rank of A^T. So the range and row space of A have the same dimension.

In particular, rank A = # dependent variables
nullity A = # independent ——
$\Big\}$ $n = \#\text{cols} A = \text{rank}(A) + \text{null}(A)$

Example above: 4 = 2+2.

Thm 4: A system $A\underline{x} = \underline{b}$ is consistent if and only if $\text{rank}(A) = \text{rank}(A|b)$

Proof: The system is consistent if and only if $\underline{b}$ in $Sp(\text{col}_1 A, \dots, \text{col}_n A) = \mathcal{R}(A)$

so $Sp(\text{col}_1 A, \dots, \text{col}_n A, b) = \mathcal{R}(A)$ & both have the same dimension.

dim = rank([A|b]). dim = rank(A).

Thm 5: A matrix A of size $n \times n$ is nonsingular if and only if $\text{rank}(A) = n$.

Proof: A is nonsingular if and only if n cols of A are l.i., if and only if $\dim \mathcal{R}(A) = n$, that is $\text{rank}(A) = n$.

(*) Q: Example above: Which pairs of $\{v_1, \dots v_4\}$ are bases for W?

Enough to check which pairs are li. A: $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_1, v_4\}$, $\{v_2, v_3\}$, $\{v_2, v_4\}$, $\{v_3, v_4\}$ (Why? Use Thm 2 (3)).

Proof of Thm 3: Write $A = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$ $\quad a_i = [a_{i1}, \cdots, a_{in}]$

Ⓘ Show $\operatorname{rank}(A) \leq \operatorname{rank}(A^T)$:

Write $k = \operatorname{rank}(A^T)$ so $W = Sp(a_1, \cdots, a_m)$ has $\dim W = k$, W is a subspace of $\mathbb{R}^n$. By Thm 2 (2): $m \geq k$. Fix a basis $B = \{w_1, \cdots, w_k\}$ of W. Write each a_i as a linear combination of B:

$$a_1^T = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{bmatrix} = \alpha_{11} \begin{bmatrix} w_{11} \\ w_{12} \\ \vdots \\ w_{1n} \end{bmatrix} + \cdots + \alpha_{1k} \begin{bmatrix} w_{k1} \\ w_{k2} \\ \vdots \\ w_{kn} \end{bmatrix}$$

$$\vdots$$

$$a_m^T = \begin{bmatrix} \alpha_{m1} \\ a_{m2} \\ \vdots \\ a_{mn} \end{bmatrix} = \alpha_{m1} \begin{bmatrix} w_{11} \\ w_{12} \\ \vdots \\ w_{1n} \end{bmatrix} + \cdots + \alpha_{mk} \begin{bmatrix} w_{k1} \\ \vdots \\ w_{kn} \end{bmatrix}$$

Take the j^{th} component of each equation:

$$\begin{cases} a_{1j} = \alpha_{11} w_{1j} + \cdots + \alpha_{1k} w_{kj} \\ \vdots \\ a_{mj} = \alpha_{m1} w_{1j} + \cdots + \alpha_{mk} w_{kj} \end{cases}$$

so $\begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} = \operatorname{col}_j A = w_{1j} \begin{bmatrix} \alpha_{11} \\ \vdots \\ \alpha_{m1} \end{bmatrix} + \cdots + w_{kj} \begin{bmatrix} \alpha_{1k} \\ \vdots \\ \alpha_{mk} \end{bmatrix}$ for each j

so $R(A)$ is a subset of $Sp\left(\begin{bmatrix} \alpha_{11} \\ \vdots \\ \alpha_{m1} \end{bmatrix}, \cdots, \begin{bmatrix} \alpha_{1k} \\ \vdots \\ \alpha_{mk} \end{bmatrix}\right) = W$

So $\operatorname{rank}(A) = \dim R(A) \leq \dim W \leq k := \operatorname{rank}(A^T)$.

Ⓘⓘ Use $(A^T)^T = A$ to conclude $\operatorname{rank}(A^T) \underset{\text{by Ⓘ applied to } A^T.}{\leq} \operatorname{rank}((A^T)^T) = \operatorname{rank}(A)$

We get $\operatorname{rank} A \leq \operatorname{rank} A^T \leq \operatorname{rank} A$ so $\operatorname{rank} A = \operatorname{rank} A^T$ as we wanted to show.