Lecture XVII §5.1-2 Vector Spaces

§1 Motivation:

So far: constructed $\mathbb{R}^n$ & (vector) subspaces of $\mathbb{R}^n$:

(1) $W = Sp(\{v_1, \ldots, v_r\})$

(2) $\mathcal{N}(A)$ for an $m \times n$ matrix A ⇝ sum of 2 solutions to a homog eqn is again a solution;

(3) $\mathcal{R}(A)$ for an $n \times m$ " A (special case of (1): v's = cols(A))

(4) RowSpace(A) for an $m \times n$ " A (——— : v's = rows (A))

Properties:
- $\mathbf{0}$ in W
- We can add two vectors of W (using + in $\mathbb{R}^n$) & remain in W
- We can scalar multiply a vector of W with any scalar (using the scalar product operation in $\mathbb{R}^n$) & remain in W.
- + & scalar multiplication have nice algebraic properties (inherited from $\mathbb{R}^n$)

• These properties can be translated to other spaces:

Ex 1: A set of solutions to a differential equation $y'' + a_{(t)}\, y' + c_{(t)}\, y = 0$.

For example: $y'' - y = 0$

Solutions: $y_{1(t)} = e^{t}$, $y_{2(t)} = e^{-t}$ & any linear combination $y_{(t)} = c_1 e^{t} + c_2 e^{-t}$ for any c_1, c_2.

These 2 solutions are linearly independent:

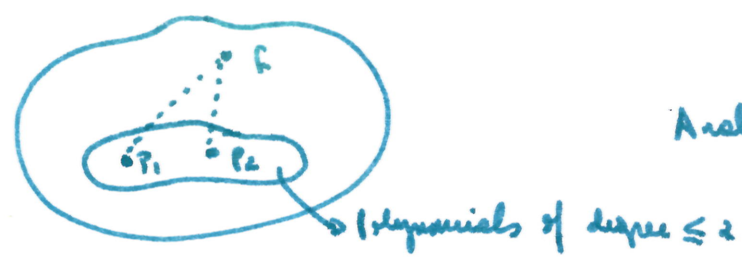
$c_1 e^{t} + c_2 e^{-t} = 0$. Evaluate at $t=0$: $c_1 + c_2 = 0$, so $c_1 = -c_2$

. " " $t = 1$ $c_1 \underbrace{(e - e^{-1})}_{\neq 0} = 0$ forces $c_1 = 0$.

So the only solution (c_1, c_2) is the trivial one.

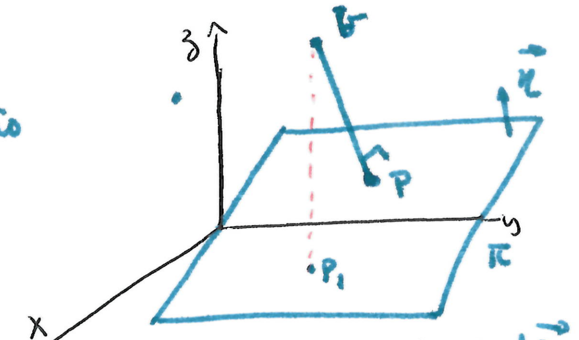
Ex 2: Approximate continuous functions by polynomials of degree ≤ 2 (3 or n)

↳ Notion of distance & find a polynomial which is closest to f.



• Minimize dist(f, P) for P in the "space of polynomials of degree 2"

Analogous to



$v - P$ is a multiple of $\vec{n}$

$v - P$ is perpendicular two the 2 directions spanning P_π.

§2 Vector Spaces:

A general vector space consists of a set of elements (called vectors) $\mathbb{V}$ & a set of scalars S ($\mathbb{R}$ or $\mathbb{C}$ for us) with 2 operations:

(1) Addition $+ : V \times V \longrightarrow V$ for v, w in $\mathbb{V}$, gives a new vector $v+w$ in $\mathbb{V}$

(2) Scalar multiplication: $\cdot : S \times V \longrightarrow V$ for v in $\mathbb{V}$ & α in S, gives a new vector $(\alpha \cdot v)$ in $\mathbb{V}$.

Addition & scalar multiplication have nice algebraic properties.

Eg: • $\mathbb{R}^n$
• $\mathbb{R}^{n \times m}$ = matrices of size $n \times m$.
• solutions to a homogeneous system ...

Formal definition: A set of elements $\mathbb{V}$ is a vector space over $\mathbb{R}$ if we can define addition & scalar multiplication on $\mathbb{V}$ & the following properties hold for any u, v, w in $\mathbb{V}$ and a, b scalars in $\mathbb{R}$:

• Closure Properties

(C1) $u+v$ is a vector in V

(C2) av is a vector in V

• Properties of Addition:

(A1) $u+v = v+u$ [Commutativity]

(A2) $u+(v+w) = (u+v)+w$ [Associative]

(A3) There is a vector $\mathbb{0}$ in V such that $v + \mathbb{0} = \mathbb{0}$ for all v in $\mathbb{V}$. [Addition Identity]

(A4) Given v in $\mathbb{V}$ there is a vector $-v$ in $\mathbb{V}$ such that $v+(-v) = \mathbb{0}$ [Addition Inverse]

[Note: $-v = (-1)\cdot v$]

• Properties of Scalar multiplication:

(M1) $a(bv) = (ab)v$ [Commutativity]

(M2) $a(u+v) = au + av$ [Assoc.]

(M3) $(a+b)v = av + bv$ [Assoc.]

(M4) $1 \cdot v = v$ for all v in $\mathbb{V}$.

Examples ⓪ $\mathbb{V} = \{\mathbb{0}\}$

① (2×3) matrices: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ • $-A = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}$ • $\mathbb{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

• $\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \end{bmatrix}$ All 10 properties hold.

② {Polynomials of degree ≤ 2} $= \mathcal{P}_2$

$ax^2 + bx + c$ for a, b, c in $\mathbb{R}$.

$P_1 = a_1 x^2 + b_1 x + c_1$
$P_2 = a_2 x^2 + b_2 x + c_2$
$\rightsquigarrow P_1 + P_2 = (a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_2)$ ✓
$\alpha P_1 = (\alpha a_1)x^2 + (\alpha b_1)x + (\alpha c_1)$ ✓

- $+$ of functions satisfies (A1), (A2).
- $\mathbb{0} = 0 \cdot x^2 + 0x + 0$ in $\mathbb{V}$.
- $-P = (-a_1)x^2 + (-b_1)x + (-c_1)$ in $\mathbb{V}$
- scalar product satisfies (M1) — (M4) ✓

$\mathcal{P}_2 = Sp(x^2, x, 1)$

③ $C[a,b] = \{ f(x)$ continuous real-valued function on $[a,b] \}$

- $+$ of cont. functions is continuous
- $\mathbb{0} =$ constant function 0 is ———
- $(\alpha f)(x) = \alpha(f(x))$ is continuous, $(-f)(x) = -f(x)$ is cont.
- A1, A2, M1 — M4 hold.

④ $\mathbb{W} = \{ P(x)$ in $\mathcal{P}_2 : P'(0) = 0 \}$ <u>subspace</u> of $\mathcal{P}_2$ $(= Sp(1, x^2))$

Why? $P(x) = ax^2 + bx + c$
$P'(x) = 2ax + b$ $\quad P'(0) = b = 0$ $\}$ so $P(x) = ax^2 + c$ (no degree 1 term)

NTS
- P, Q in $\mathbb{W}$, then $P + Q$ in $\mathbb{W}$ because $(P+Q)'(0) = P'(0) + Q'(0) = 0 + 0 = 0$.
- αP in $\mathbb{W}$ because $(\alpha P)'(0) = \alpha P'(0) = \alpha \cdot 0 = 0$ ✓
- Properties (A1) — (M4) are inherited from $\mathcal{P}_2$.

⑤ $\mathbb{W} = \{ 2 \times 2$ singular matrices$\}$ is <u>NOT</u> a vector space

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ in $\mathbb{W}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in $\mathbb{W}$ <u>but</u> $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ nonsing, so <u>not</u> in $\mathbb{W}$.

⑥ $\mathbb{W} = \mathbb{R}^2$ with funny addition: $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 + 1 \\ u_2 + v_2 + 1 \end{bmatrix}$, usual scalar mult

- (C1) & (C2) hold
- (A1) ✓ (A2) ✓
- (A3): $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ if and only if $\underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{= \mathbb{0}} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ $\rightsquigarrow$ Additive Identity
- (A4) $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 + 1 \\ u_2 + v_2 + 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ if and only if $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -u_1 \\ -u_2 \end{bmatrix} = -u$

• (M1) & (M4) ✓

• (M2) fails $a=2$ $u=\begin{bmatrix}1\\1\end{bmatrix}$, $v=\begin{bmatrix}0\\1\end{bmatrix}$

$$2(u \oplus v) = 2\left(\begin{bmatrix}1\\1\end{bmatrix} \oplus \begin{bmatrix}0\\1\end{bmatrix}\right) = 2\left(\begin{bmatrix}1+0+1\\1+1+1\end{bmatrix}\right) = 2\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix}$$

$$2u \oplus 2v = \begin{bmatrix}2\\2\end{bmatrix} \oplus \begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}2+0+1\\2+2+1\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix} \quad \neq$$

• (M3) Also fails: $a=1$ $b=2$ $v=\begin{bmatrix}1\\1\end{bmatrix}$

$$(a+b)\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix} \quad \& \quad a\begin{bmatrix}1\\1\end{bmatrix} \oplus b\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1+2+1\\1+2+1\end{bmatrix} = \begin{bmatrix}4\\4\end{bmatrix} \rightarrow \text{differ!}$$

So $\mathbb{W}$ is not a vector space.

Properties Cancellation Laws:

(1) if $u+v=u+w$, then $v=w$ (Why? Add $-u$ on both sides

(2) if $v+u=w+u$, " $v=w$ (A4) + (A1))

Theorem: (1) The zero vector $\mathbb{0}$ is unique.

(2) The additive inverse is unique & $-v=(-1)\cdot v$.

(3) $0 \cdot v = \mathbb{0}$

(4) $\alpha \cdot \mathbb{0} = \mathbb{0}$ for every α scalar

(5) If $\alpha v = \mathbb{0}$, either $\alpha = 0$ or $v = \mathbb{0}$.

(Why? If $\alpha \neq 0$, multiply by $\frac{1}{\alpha}$: $\mathbb{0} = \frac{1}{\alpha}(\alpha v) = \left(\frac{1}{\alpha}\alpha\right)v = 1 \cdot v = v$)