Lecture XVIII: §5.3 Vector Subspaces

Recall: Modelled by $\mathbb{R}^n$ we defined the notion of an (abstract) vector space over $\mathbb{R}$ as a set of elements $\mathbb{V}$ (called vectors) together with 2 operations

- Addition in $\mathbb{V}$
- scalar multiplication by $\mathbb{R}$

satisfying 10 properties:

- Closure: (1) v, w in $\mathbb{V}$ then $v+w$ in $\mathbb{V}$
  (2) v in $\mathbb{V}$, α in $\mathbb{R}$, then $\alpha \cdot v$ in $\mathbb{V}$
- Neutral element $\mathbb{O}$ in $\mathbb{V}$ ($v + \mathbb{O} = v$ for any v in $\mathbb{V}$)
  [which is unique]
- Inverse for addition: given v in $\mathbb{V}$ we have $v + \underbrace{(-1)\cdot v}_{=:-v} = \mathbb{O}$. (Addition inverse) which is UNIQUE
- 6 more algebraic properties ($+$ & $\cdot$ interact nicely).

- In $\mathbb{R}^n$, we also had a notion of subspace (e.g. null space, row space, range, lines, planes in $\mathbb{R}^3$ through $(0,0,0)$)

TODAY: Study subspaces of an abstract vector space.

Definition: If $\mathbb{V}$ and $\mathbb{W}$ are vector spaces, $\mathbb{W}$ is a subset of $\mathbb{V}$ & addition & scalar product in $\mathbb{V}$ agrees with $\mathbb{W}$, then $\mathbb{W}$ is a subspace of $\mathbb{V}$

- Just as it happened with $\mathbb{R}^n$, we can check this through 3 conditions:

Thm: If $\mathbb{W}$ is a subset of $\mathbb{V}$ & $\mathbb{V}$ is V.Sp., $\mathbb{W}$ is a subset of $\mathbb{V}$ if and only if:

(S1) The $\mathbb{O}$ zero vector of $\mathbb{V}$ lies in $\mathbb{W}$.

(S2) given u, v in $\mathbb{W}$, then $u+v$ lies in $\mathbb{W}$ ↳ addition in $\mathbb{V}$

(S3) Given u in $\mathbb{W}$, α in $\mathbb{R}$, then $\alpha \cdot u$ lies in $\mathbb{W}$.

Why? Remaining 7 properties for $\mathbb{W}$ to be a vector space are inherited from $\mathbb{V}$. □

§1 Examples:

(I) Last time: $M_{2\times 3} = \{ (2\times 3) \text{ matrices} \}$ is a vector space (addition & scalar mult in each entry $\mathbb{O} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$)

Claim: $\mathbb{W}_1 = \left\{ \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \end{pmatrix} : a_{11}, a_{13}, a_{22} \text{ in } \mathbb{R} \right\}$ is a subspace.

(S1) $\mathbb{O}$ lies in $\mathbb{W}$ ✓

(S2) $\underbrace{\begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \end{pmatrix}}_{\text{"A"}} + \underbrace{\begin{pmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{22} & 0 \end{pmatrix}}_{\text{"B"}} = \begin{pmatrix} a_{11}+b_{11} & 0 & a_{13}+b_{13} \\ 0 & a_{22}+b_{22} & 0 \end{pmatrix}$ in $\mathbb{W}$ ✓

(S3) $\alpha \cdot \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & 0 & \alpha a_{13} \\ 0 & \alpha a_{22} & 0 \end{pmatrix}$ in $\mathbb{W}$ ✓

Claim: $\mathbb{W}_2^* = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad a_{11}a_{22} - a_{21}a_{12} = 0 \right\}$ is NOT a subspace

(S1) ✓ (S3) $\alpha A = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \end{pmatrix}$ $\quad (\alpha a_{11})(\alpha a_{22}) - (\alpha a_{12})(\alpha a_{21})$

$= \alpha^2 \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{=0} = 0$ ✓

(S2) fails: $\left.\begin{array}{l} A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}\right\}$ but $A+B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ & $1 \cdot 1 - 0 \cdot 0 = 1 \neq 0$

so $A+B$ NOT in $\mathbb{W}$.

(II) Last Time: $C_{[a,b]} = \{ f : [a,b] \longrightarrow \mathbb{R} \text{ continuous} \}$ $\quad \bullet\ \mathbb{O}: \mathbb{O}_{(x)} = 0$ for all x

$\bullet\ (\alpha f)_{(x)} = \alpha f_{(x)}$

$\bullet\ (f+g)_{(x)} = f_{(x)} + g_{(x)}$

Claim $\mathbb{W}_3 = \left\{ f \text{ in } C_{[a,b]} : \int_a^b f_{(x)}\, dx = 0 \right\}$ is a subspace

Recall: f continuous, then integrable

(S1) $\mathbb{O}$ lies in $\mathbb{W}_3$ because $\int_a^b 0\, dx = 0$ ✓

(S2) f, g in $\mathbb{W}_3$, $\int_a^b (f+g)_{(x)}\, dx = \int_a^b f_{(x)}\, dx + \int_a^b g_{(x)}\, dx = 0 + 0 = 0$ ✓

(S3) $\int_a^b (\alpha f)_{(x)}\, dx = \int_a^b \alpha f_{(x)}\, dx = \alpha \int_a^b f_{(x)}\, dx = \alpha \cdot 0 = 0$ ✓

Claim: $\mathbb{W}_4 = \{ f \text{ in } C_{[0,2]} : f'_{(1)} = 0 \}$
[a=0, b=2]

(S1) $\mathbb{O}'_{(1)} = 0$ so $\mathbb{O}$ in $\mathbb{W}_4$

(S2) $(f+g)'_{(1)} = f'_{(1)} + g'_{(1)} = 0 + 0$ in $\mathbb{W}_4$ if f, g are in $\mathbb{W}_4$

(S3) $(\alpha f)'_{(1)} = \alpha f'_{(1)} = \alpha \cdot 0 = 0$ in $\mathbb{W}_4$ if f in $\mathbb{W}_4$.

§2 Spanning Sets

We use the same definitions as in $\mathbb{R}^n$

Def A vector v in $\mathbb{V}$ is a linear combination of vectors $v_1, \dots, v_r$ in $\mathbb{V}$ if $v = a_1 v_1 + \cdots + a_r v_r$ for some scalars $a_1, \dots, a_r$.

Example (last time) $\mathcal{P}_2 = \{\text{polynomials of degree} \leq 2\} = \{ a x^2 + bx + c ;\ a,b,c \text{ in } \mathbb{R} \}$

Any polynomial $\mathcal{P}_2$ is a linear combination of $x^2, x, 1$ ("vectors" in $\mathcal{P}_2$)

Write $\mathcal{P}_2 = \mathrm{Sp}(x^2, x, 1)$.

Def: If we have $\{v_1, \dots, v_r\}$ in $\mathbb{V}$ and any v in $\mathbb{V}$ is a l.comb of $v_1, \dots, v_r$, we say these r vectors <u>span</u> $\mathbb{V}$.

Write $\mathbb{V} = Sp(v_1, \dots, v_r)$

Q: How to find spanning sets? <u>Warning</u>: They not always exist! (eg $C_{[0,1]}$)

<u>Examples</u> (revisited)

- For $\mathbb{W}_1$: $A = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \end{pmatrix} = a_{11}\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=E_{11}} + a_{22}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{=E_{22}} + a_{13}\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{=E_{13}}$

  so $\mathbb{W}_1 = Sp(E_{11}, E_{22}, E_{13})$.
- $C_{[a,b]}$ has no spanning set. Neither ~~of~~ $\mathbb{W}_3$, $\mathbb{W}_4$
- <u>Last Time</u>: $\mathbb{W} = \{P \text{ in } \mathcal{P}_2 : P'(0) = 0\} = \{a x^2 + c : a, c \text{ in } \mathbb{R}\}$ is a v.space (subspace of $\mathcal{P}_2$) $= Sp(x^2, 1)$.

Just as it happened with $\mathbb{R}^n$, the following statement holds:

<u>Thm 2</u>: If $\mathbb{V}$ is a vector space & $\{v_1, \dots, v_r\}$ are vectors in $\mathbb{V}$, then $\mathbb{W} = Sp(v_1, \dots v_r)$ is a subspace of $\mathbb{V}$.

[Proof is the same as we did in <u>Lecture XIV</u>.]

<u>Example</u>: $W = \{\underbrace{\text{Symmetric}}_{A^T = A} \ 3\times 3 \text{ matrices}\} = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \ a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33} \text{ in } \mathbb{R} \right\}$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \rightsquigarrow \boxed{\begin{matrix} a_{21} = a_{12} \\ a_{31} = a_{13} \\ a_{23} = a_{32} \end{matrix}} \text{ defining equations}$$

$$A = a_{11}\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=E_{11}} + a_{12}\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=E_{12}+E_{21}} + a_{13}\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{=E_{13}+E_{31}} + a_{22}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=E_{22}}$$
$$+ a_{23}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{=E_{23}+E_{32}} + a_{33}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{=E_{33}}$$

$$W = Sp(E_{11}, E_{12}+E_{21}, E_{22}, E_{13}+E_{31}, E_{23}+E_{32}, E_{33})$$

<u>Example</u>: $W = \{\underbrace{\text{skew-symmetric}}_{A^T = -A} \ 3\times 3 \text{ matrices}\}$

$$-A = -\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \rightsquigarrow \text{defining equations}$$

- $a_{11} = a_{22} = a_{33} = 0$
- $a_{12} = -a_{21}$ $\quad a_{23} = -a_{32}$
- $a_{13} = -a_{31}$

$$W = \left\{ \begin{pmatrix} 0 & -a_{21} & -a_{31} \\ a_{21} & 0 & -a_{32} \\ a_{31} & a_{32} & 0 \end{pmatrix} \quad a_{21}, a_{31}, a_{32} \text{ in } \mathbb{R} \right\}$$

$$W = Sp \left\{ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

- Minimal spanning set = spans & no proper subset spans (ALL sets are minimal spanning sets)

This will naturally lead to defining <u>bases</u> for abstract vector spaces.