Lecture XIX §5.4 Linear Independence, Bases & Coordinates

Last Time:
- Defined subspace of an abstract vector space $\mathbb{V}$
- **Example**: $Sp(v_1, \ldots, v_r)$ as a subspace of $\mathbb{V}$ obtained as all linear combinations of the vectors $v_1, \ldots, v_r$ from $\mathbb{V}$.
- Not all subspaces have a (finite) spanning set (eg $C[0,1]$).

TODAY: linear independence, bases & coordinates of vectors with respect to a basis.

§1 Linear independence:

Def: Fix a vector space $\mathbb{V}$ & $v_1, \ldots, v_r$ vectors in $\mathbb{V}$. The set $\{v_1, \ldots, v_r\}$ is <u>linearly dependent</u> if there are scalars $\alpha_1, \ldots, \alpha_r$ NOT all zero, satisfying

$$(*) \qquad \alpha_1 v_1 + \cdots + \alpha_r v_r = \mathbb{O} \to \text{zero vector in } \mathbb{V}.$$

- The set $\{v_1, \ldots, v_r\}$ is <u>linearly independent</u> if the only scalars solving $(*)$ are $\alpha_1 = \cdots = \alpha_r = 0$. (trivial solution)

Note: Same definition as l.i in $\mathbb{R}^n$ ⇝ same methods to decide l.i or not!

METHOD 1: Solve a homogeneous system:

Eg 1 (last time): $\mathbb{W} = \left\{ \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \end{pmatrix} \; a_{11}, a_{13}, a_{22} \text{ in } \mathbb{R} \right\} = Sp\left(\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{v_2}, v_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right)$

Claim: 3 vectors are l.i.

$\boxed{x_i = \text{unknown scalars}}$

Why? $x_1 v_1 + x_2 v_2 + x_3 v_3 = \mathbb{O} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} x_1 & 0 & x_2 \\ 0 & x_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \text{system } \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases} \text{ has only the trivial soln!}$$

Eg 2: $\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 3 & 0 & 5 \\ 0 & 3 & 0 \end{bmatrix}}_{v_3}, \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{v_4}$ are l.d.

$$x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + x_2 + 3x_3 & 0 & x_1 + 2x_2 + 5x_3 + x_4 \\ 0 & x_1 + x_2 + 3x_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

⇝ System $\begin{cases} x_1 + x_2 + 3x_3 = 0 \\ x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 + 5x_3 + x_4 = 0 \end{cases}$) same equation!

⇝ (Augmented) matrix $[A|0]$ make it into REF: $\begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 5 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - R_1]{} \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ REF

So $x_1 = -x_3 + x_4$
$x_2 = -2x_3 - x_4$

general form: $\underline{x} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

• 2 dependencies: $\begin{cases} -v_1 - 2v_2 + v_3 = 0 \\ v_1 - v_2 + v_4 = 0 \end{cases}$ → $\{v_1, v_2, v_3\}$ are l.d., and hence so is $\{v_1, v_2, v_3, v_4\}$

⇝ v_3, v_4 are both in $Sp\{v_1, v_2\}$, so $Sp\{v_1, v_2, v_3, v_4\} = Sp\{v_1, v_2\}$.

Aside: To check if $\{v_1, v_3, v_4\}$ is l.i., we set $x_2 = 0$ (v_2 is not in the equation)

But $x_2 = -(x_4 + 2x_3) = 0$ (from the general form) & $\underline{x} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + (-2x_3) \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

so $-3v_1 + v_3 + v_4 = 0$ ⇝ so they are l.d.

METHOD 2: For subspaces of functions, we can evaluate at values of x (as many as vectors we have)

Example 3: Show that $\{1, x, x^2\}$ in $\mathcal{P}_2 := \{$degree ≤ 2 polynomials in $x\}$ is l.i.

Here: 3 vectors: $v_1 = 1$
$v_2 = x$
$v_3 = x^2$

write $\underbrace{a + bx + cx^2}_{\text{scalars: } a,\, b,\, c} = 0 =$ constant function $= 0$.

We evaluate at 3 different values of x:

• At $x = 0$: $a + 0 + 0 = 0$ so $\boxed{a = 0}$
• At $x = 1$: $0 + b + c = 0$ ⇝ $b = -c$
• At $x = -1$: $0 - b + c = 0$ ⇝ $b = c$

so $\boxed{b = 0 \ \& \ c = 0}$. So the 3 vectors are l.i.

Example 4: $\{1, (x+1)^2, (x-1)^2, x^2\}$ is l.d.

$a + b(x+1)^2 + c(x-1)^2 + dx^2 = 0$

At $x = 1$: $a + 4b + d = 0$
At $x = -1$: $a + 4c + d = 0$
At $x = 0$: $a + b + c = 0$
At $x = 2$: $a + 9b + c + 4d = 0$

⇝ solve the system for a, b, c, d (4 unknowns)

$$\begin{bmatrix} 1 & 4 & 0 & 1 \\ 1 & 0 & 4 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 9 & 1 & 4 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}]{} \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & -4 & +4 & 0 \\ 0 & -3 & 1 & -1 \\ 0 & 5 & 1 & 3 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2/-4 \\ R_3 \to R_3 + 3R_2 \\ R_4 \to R_4 - 5R_2}]{} \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 6 & 3 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3/-2 \\ R_4 \to R_4 - 6R_3}]{} \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\substack{R_2 \to R_2 + R_3 \\ R_1 \to R_1 - 4R_2}]{} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so $a = d$
$b = \frac{1}{2}d$
$c = \frac{1}{2}d$

⇝ dependence relation: $\boxed{0 = 1 + \frac{1}{2}(x+1)^2 + \frac{1}{2}(x-1)^2 + \frac{x^2}{2}}$

§2 Vector-Space Bases:

Def: Let $\mathbb{V}$ be a vector space. A set $B = \{v_1, \dots, v_r\}$ is a basis for $\mathbb{V}$ if

(1) B is a spanning set for $\mathbb{V}$.

(2) B is li (so it's a minimal spanning set)

Examples $M_{2\times 3} = Sp\left(\underset{E_{11}}{\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}}, \underset{E_{12}}{\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}}, \underset{E_{13}}{\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}}, \underset{E_{21}}{\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}}, \underset{E_{22}}{\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}}, \underset{E_{23}}{\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}} \right)$

In general: $M_{m\times n}$ has a basis of size $m\cdot n$ (E_{ij} = matrix with 0's everywhere except (i,j)-entry $= 1$)
= matrices of size $m \times n$.

· **Example 1** (revisited) $\{E_{11}, E_{13}, E_{22}\}$ is a basis for $\left\{\begin{pmatrix}a&0&b\\0&c&0\end{pmatrix} : a, b, c \text{ in } \mathbb{R}\right\} = \mathbb{W}$

· **Example 3** (rev.) $\mathbb{P}_2$ has a basis $\{1, x, x^2\}$

Q: How to find a basis from a spanning set?

ALGORITHM: Starting from $B = \overset{\{v_1, \dots, v_r\}}{\text{spanning set}}$, remove vectors from dependence relations coming from "independent unknowns" in solns to $x_1 v_1 + \dots + x_r v_r = 0$ in $\mathbb{V}$.

§3 Coordinate Vectors:

Bases are minimal spanning sets, and as in the case of $\mathbb{R}^n$, they can be used to give coordinates to their spans.

Theorem 1: Given a vector space $\mathbb{V}$ with basis $B = \{v_1, \dots, v_p\}$, the representation of each v in $\mathbb{V}$ as a lin. comb of B is unique; meaning the scalars in

vectors

$$v = a_1 v_1 + \dots + a_p v_p$$

scalars

are unique. We call them the coordinates of v with respect to v and write

$$[v]_B := \begin{array}{c} \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} \end{array}\begin{array}{l} v_1 \\ \vdots \\ v_p \end{array} = \text{a vector in } \mathbb{R}^p. \quad (\text{order of } B \text{ matters!})$$

Proof: Same as we did for $\mathbb{R}^n$. (see Lecture XV)

Consequence: We can identify $\mathbb{V}$ with $\mathbb{R}^p$ via $[v]_B \longleftrightarrow v$.

Example 1 (again): $\left[\begin{pmatrix}a&0&b\\0&c&0\end{pmatrix}\right]_B = \begin{bmatrix}a\\b\\c\end{bmatrix}\begin{array}{l}E_{11}\\E_{13}\\E_{22}\end{array}$ if $B = \{E_{11}, E_{13}, E_{22}\}$ & $\mathbb{W} \longleftrightarrow \mathbb{R}^3$

Example 2 $v = \underline{\underline{a}}\,1 + \underline{\underline{b}}\,x + \underline{\underline{c}}\,x^2$ in $\mathcal{P}_2$ $\rightsquigarrow$ $[v]_B = \begin{array}{cc} \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \begin{array}{c} 1 \\ x \\ x^2 \end{array} \end{array}$ $B = \{1, x, x^2\}$

scalars vectors

$[v]_{B'} = \begin{array}{cc} \begin{bmatrix} c \\ b \\ a \end{bmatrix} & \begin{array}{c} x^2 \\ x \\ 1 \end{array} \end{array}$ $B' = \{x^2, x, 1\}$

Lemma: Identification via coordinates w.r.t a basis B behaves well with respect to addition & scalar mult. in $\mathbb{V} = Sp(B)$.

(1) $[0]_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{R}^p$

(2) $[v + w]_B = [v]_B + [w]_B$ for any v, w in $\mathbb{V}$

(3) $[\alpha v]_B = \alpha [v]_B$ for v in $\mathbb{V}$, α in $\mathbb{R}$

Theorem 2: Given a vector space $\mathbb{V}$ with basis $B = \{v_1, \ldots, v_p\}$, an a set of vectors $W = \{w_1, \ldots, w_m\}$ in $\mathbb{V}$, write $T = \{[w_1]_B, \ldots, [w_m]_B\}$ in $\mathbb{R}^p$.

(1) A vector v in $\mathbb{V}$ lies in $Sp(W)$ if and only if $[v]_B$ lies in $Sp(T)$

[$v = a_1 w_1 + \cdots + a_m w_m$ if and only if $[v]_B = a_1 [w_1]_B + \cdots + a_m [w_m]_B$ so they use the SAME scalars!]

(2) The set W is lin ind if and only if T is l.i. in $\mathbb{R}^p$.

Consequence: W is a basis for $\mathbb{V}$ if and only if T is a basis for $\mathbb{R}^p$.

In particular, all basis of $\mathbb{V}$ have the same number of vectors, which we define as the dimension of $\mathbb{V}$