Lecture XXI: §3.7 Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

GOAL: Study functions between subspaces of $\mathbb{R}^n$ & $\mathbb{R}^m$ that respect the vector space structures on both sides (addition & scalar multiplication)

§1. Examples ① $F: \mathbb{R}^3 \longrightarrow \mathbb{R}$ $\quad F(\underline{x}) = F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = x_1 + 5x_3 = x_1 + 0\cdot x_2 + 5x_3$ $\quad$ linear expression in x_1, x_2, x_3

F linear:

- $\boxed{F(\underline{u}+\underline{v})} = F\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = F\left(\begin{bmatrix} \overbrace{u_1+v_1}^{=x_1} \\ u_2+v_2 \\ \underbrace{u_3+v_3}_{=x_3} \end{bmatrix}\right) = u_1+v_1+5(u_3+v_3)$
$= (u_1+5u_3)+(v_1+5v_3)$
$= \boxed{F(\underline{u}) + F(\underline{v})}$

- $\boxed{F(\alpha\underline{u})} = F\left(\begin{bmatrix} \alpha u_1 \\ \alpha u_2 \\ \alpha u_3 \end{bmatrix}\right) = \alpha u_1 + 5(\alpha u_3)$
$= \alpha(u_1+5u_3) = \boxed{\alpha F(\underline{u})}$

α scalar in $\mathbb{R}$
u in $\mathbb{R}^3$

Notice: • $F(\underline{x}) = \begin{bmatrix} 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

• We can restrict F to a line L or a plane P through the origin in $\mathbb{R}^3$, and get 2 functions $F: \underset{\text{line}}{L} \longrightarrow \mathbb{R}$, $F: \underset{\text{plane}}{P} \longrightarrow \mathbb{R}$ linear.

② • We can get a function $G: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by assigning $F_1: \mathbb{R}^3 \longrightarrow \mathbb{R}$, $F_2: \mathbb{R}^3 \longrightarrow \mathbb{R}$ (2 linear functions)

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longrightarrow \begin{bmatrix} F_1(\underline{x}) \\ F_2(\underline{x}) \end{bmatrix}$

to each of the 2 coordinates in the target $\mathbb{R}^2$. The result will be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.

Eg: $F_1(\underline{x}) = \begin{bmatrix} 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$F_2(x) = 3x_1 - 7x_2 + 8x_3 = \begin{bmatrix} 3 & -7 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Then $G(\underline{x}) = \begin{bmatrix} x_1 + 5x_3 \\ 3x_1 - 7x_2 + 8x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 5 \\ 3 & -7 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\rightsquigarrow$ multiplication by a (2×3)-matrix gives $G: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ linear transformation

$G\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1^{st}$ column of A

$G\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -7 \end{bmatrix} = 2^{nd}$ ______

$G\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 8 \end{bmatrix} = 3^{rd}$ ______

So: values on the canonical basis e_1, e_2, e_3 for $\mathbb{R}^3$ determine the matrix A

& so $G\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = G\left(x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \overset{\boxed{G \text{ linear}}}{=} x_1 G\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + x_2 G\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + x_3 G\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

<u>Example 2</u>: $G : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ $\quad G(\underline{x}) = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 0 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 5x_3 \\ 2x_1 + 10x_3 \end{bmatrix}$

"scalar"

<u>Note</u>: $G(x) = \begin{bmatrix} x_1 + 5x_3 \\ 2(x_1 + 5x_3) \end{bmatrix} = (x_1 + 5x_3) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

So points in the image live in the line L with direction $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

⇝ We can write $G : \mathbb{R}^3 \longrightarrow L$ (image is in the subspace L of $\mathbb{R}^2$)

§2 General definition:

<u>Def</u>: Let $\mathbb{V}$ be a subspace of $\mathbb{R}^n$ & $\mathbb{W}$ be a subspace of $\mathbb{R}^m$. Consider a function T from $\mathbb{V}$ to $\mathbb{W}$:

$$T : \mathbb{V} \longrightarrow \mathbb{W}$$

We say T is a <u>linear transformation</u> if for all $\underline{u}$ & $\underline{v}$ in $\mathbb{V}$ & all scalars α in $\mathbb{R}$ we have:

(1) $T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$

(2) $T(\alpha \underline{u}) = \alpha T(\underline{u})$

<u>Remark</u> (later) Every linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ corresponds to multiplication by a matrix A of size $m \times n$, namely $T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

⚠ We will give a similar definition for abstract vector spaces, but the remark no longer holds necessarily, only works when the spaces involve have finite bases (hence we can identify them with subspaces of $\mathbb{R}^n$ & $\mathbb{R}^m$ via coordinates with respect to a basis $\mathcal{B}$ of $\mathbb{V}$ & a basis $\mathcal{B}'$ of $\mathbb{W}$ & get $T : \mathbb{V} \longrightarrow \mathbb{W}$ linear)

<u>Non-example</u>: $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\quad F(\underline{x}) = \begin{bmatrix} x_1 - x_2 + 1 \\ x_2 \\ -x_2 \end{bmatrix}$ is <u>NOT</u> linear

$F\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $F\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $F\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

<u>BUT</u> $F\begin{bmatrix} 1 \\ 0 \end{bmatrix} + F\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

What went wrong? The <u>non-linear factor</u> in the first coordinate!

<u>Non-example 2</u>: $F : \mathbb{R} \to \mathbb{R}$ $\quad F(x) = e^x$ is non-linear.

$F(0) = 1$, $F(1) = e$ $\quad$ But $F(0) + F(1) = 1 + e \neq e = F(0+1)$

Prop: $F:\mathbb{R}\longrightarrow\mathbb{R}$ linear if and only if $F(x)=ax$ for a fixed scalar $\underline{\underline{a}}$.

$\hookrightarrow$ clearly linear!

$$\begin{cases} F(x+y) = a(x+y) = F(x)+F(y) \\ F(\alpha x) = a(\alpha x) = \alpha F(x). \end{cases}$$

Why? Use Property (2) in the definition:

$$\underbrace{F(x)}_{x \text{ scalar}} = x\,\underbrace{F(1)}_{=a} = ax$$

Prop 2: $F:\mathbb{R}^n\longrightarrow\mathbb{R}$ linear if and only if $F(x)=u^Tx$ for some vector u in $\mathbb{R}^n$.

Why? We can explicitly find u!

$$F\left(\begin{bmatrix} x_1\\ \vdots \\ x_n\end{bmatrix}\right) = F\left(x_1\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}+\cdots+x_n\begin{bmatrix}0\\ \vdots\\ 0\\1\end{bmatrix}\right)$$

$$\underset{F \text{ linear}}{=} F\left(x_1\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}\right)+F\left(x_2\begin{bmatrix}0\\1\\ \vdots\\0\end{bmatrix}\right)+\cdots+F\left(x_n\begin{bmatrix}0\\ \vdots\\0\\1\end{bmatrix}\right)$$

$$= x_1\underbrace{F\left(\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}\right)}_{=u_1} + x_2\underbrace{F\left(\begin{bmatrix}0\\1\\ \vdots\\0\end{bmatrix}\right)}_{=u_2} + \cdots + x_n\underbrace{F\left(\begin{bmatrix}0\\ \vdots\\0\\1\end{bmatrix}\right)}_{=u_n}$$

$$= [u_1\ u_2\ \cdots\ u_n]\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = \underline{u}^T\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$$

Note: $\underline{u}^T = [F(e_1), \ldots, F(e_n)]$ $\rightsquigarrow$ values of F at the canonical basis of $\mathbb{R}^n$ $\{e_1,\ldots,e_n\}$ □

For a general $T:\mathbb{R}^n\longrightarrow\mathbb{R}^m$ linear, we can build it from m functions

$$\begin{cases} T_1:\mathbb{R}^n\longrightarrow\mathbb{R} \text{ linear} \\ \vdots \\ T_m:\mathbb{R}^n\longrightarrow\mathbb{R} \text{ linear}\end{cases} \rightsquigarrow \begin{array}{ll} T_1(x) = v_1^T\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} & \text{for a vector } v_1 \text{ in } \mathbb{R}^n \\ \vdots & \vdots \\ T_m(x) = v_m^T\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} & \text{———— } v_m \text{ in } \mathbb{R}^n\end{array}$$

So
$$T(\underline{x}) = \underbrace{\begin{bmatrix} v_1^T \\ \vdots \\ v_m^T\end{bmatrix}}_{m\times n \text{ matrix}}\begin{bmatrix}x_1\\ \vdots\\ x_m\end{bmatrix}$$

We conclude:

Thm: Every linear transformation $T:\mathbb{R}^n\longrightarrow\mathbb{R}^m$ is obtain as

$T(\underline{x}) = A\,x$ for an $(m\times n)$ matrix.

Moreover: 1^{st} column $(A) = T\left(\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}\right)$

$\vdots$

n^{th} ———— $(A) = T\left(\begin{bmatrix}0\\ \vdots\\0\\1\end{bmatrix}\right)$

so
$$A = \begin{array}{c} \begin{array}{ccc} \text{col } 1 & & \text{col } n \end{array} \\ \begin{bmatrix} T(e_1) & \cdots & T(e_n)\end{bmatrix}\end{array}$$

Notice: Only thing we use about $\{e_1 \ldots e_n\}$ is the basis property.

Exercise: Find a transformation $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $F\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\4\\3\end{bmatrix}$, $F\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\0\\5\end{bmatrix}$. Is F unique?

Sol:

$e_1 = 2\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix} \rightsquigarrow F(e_1) = 2F\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - F\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = 2\begin{bmatrix}1\\4\\3\end{bmatrix} - \begin{bmatrix}1\\0\\5\end{bmatrix} = \begin{bmatrix}1\\8\\1\end{bmatrix}$

$e_2 = \begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix} \rightsquigarrow F(e_2) = F\left(\begin{bmatrix}1\\2\end{bmatrix}\right) - F\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\5\end{bmatrix} - \begin{bmatrix}1\\4\\3\end{bmatrix} = \begin{bmatrix}0\\-4\\2\end{bmatrix}$

Conclusion: $F\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\8 & -4\\1 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\8x-4y\\x+2y\end{bmatrix}$ $\rightsquigarrow$ so it is UNIQUE.

Check: $F\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\4\\3\end{bmatrix}$ ✓, $F\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\0\\5\end{bmatrix}$ ✓

§3 Null Space & Range:

$T: \mathbb{V} \longrightarrow \mathbb{W}$ — $\mathbb{V}$ subspace of $\mathbb{R}^n$, $\mathbb{W}$ = subspace of $\mathbb{R}^m$

2 natural subspaces arising from T: Nullspace (in $\mathbb{V}$), Range (in $\mathbb{W}$)

Def: • The Null space of T is $\mathcal{N}(T) = \{v : v \text{ in } \mathbb{V} \ \& \ T(v) = \mathbb{0}\}$ — zero vector in $\mathbb{W}$

• The Range of T is $\mathcal{R}(T) = \{w : w \text{ in } \mathbb{W} \ \& \ w = T(v) \text{ for some } v \text{ in } \mathbb{W}\}$

If $\mathbb{V} = \mathbb{R}^n$, $\mathbb{W} = \mathbb{R}^m$, then $T(x) = Ax$ for some $(n \times m)$ matrix A & $\mathcal{N}(T) = \mathcal{N}(A)$ & $\mathcal{R}(T) = \mathcal{R}(A)$.

• dimension of $\mathcal{R}(T) =:$ rank (T)

• dimension of $\mathcal{N}(T) =:$ nullity (T).

Thm: If $\mathbb{V} = \mathbb{R}^n$ & $\mathbb{W} = \mathbb{R}^m$, $\boxed{\text{rank}(T) + \text{nullity}(T) = n.}$ — replace by dim $\mathbb{V}$.

Example above: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

• rank (T) = rank $(A) = 2$ $\left(A^T = \begin{bmatrix}1 & 8 & 1\\0 & -4 & 2\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0 & 5\\0 & 1 & -\frac{1}{2}\end{bmatrix} \text{ REF} \rightsquigarrow \text{rk} = 2. \text{ (2 non zero rows!)}\right)$

• nullity (T) = $A = \begin{bmatrix}1 & 0\\8 & -4\\1 & 2\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0\\0 & -4\\0 & 2\end{bmatrix} \longrightarrow \begin{array}{cc} x & y \\ \end{array}\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix} \rightsquigarrow \begin{cases}x = 0\\y = 0\end{cases}$ so the system $\begin{cases}x = 0\\8x - 4y = 0\\x + 2y = 0\end{cases}$ has unique solution $= \begin{bmatrix}0\\0\end{bmatrix}$ REF so nullity $= 0$

• $2 + 0 = 2$ $(= n)$.