# Lecture XXII: §3.7 (cont.) Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

## §1 Matrix representations:

**Recall**: A linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ (or subspaces $\mathbb{V}$ & $\mathbb{W}$ of $\mathbb{R}^n$ & $\mathbb{R}^m$, respectively) is a map $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ satisfying

(1) $T(\underline{u}+\underline{v}) = T(\underline{u}) + T(\underline{v})$ for all $\underline{u}, \underline{v}$ in $\mathbb{R}^n$

(2) $T(\alpha \underline{u}) = \alpha T(\underline{u})$ for all $\underline{u}$ in $\mathbb{R}^n$ & $\alpha$ scalar

[$T(\underline{u})$ is a vector in $\mathbb{R}^m$]

**Thm**: Any linear transf $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ equals $T(\underline{x}) = A\,\underline{x}$ for a matrix $A$ of size $m \times n$ where $A = [T(e_1) \cdots T(e_n)]$ canonical basis of $\mathbb{R}^n$.

**Note**: $A$ = coefficient matrix of the linear expressions.

Eg $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ $\quad T(\underline{x}) = \begin{bmatrix} x_1 + 2x_3 \\ x_2 - 5x_3 \end{bmatrix} \rightsquigarrow A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \end{bmatrix}$

$$\begin{array}{ccc} \shortparallel & \shortparallel & \shortparallel \\ T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) & T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) & T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) \end{array}$$

- Thm says that $T$ is completely determined by its values at the canonical basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$
- But any basis of $\mathbb{R}^n$ is as good as the canonical one.

**Prop**: Given a basis $B = \{v_1, \cdots v_n\}$ for $\mathbb{R}^n$ & vectors $w_1 \ldots w_n$ in $\mathbb{R}^m$ there is a unique linear Transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ s.t. $T(v_1) = w_1, \ldots\ldots, T(v_n) = w_n$.

**Example** (last time) $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\4\\3\end{bmatrix}$, $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\0\\5\end{bmatrix}$ $\rightsquigarrow T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x\\8x-4y\\x+2y\end{bmatrix}$. $\quad T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

We constructed $T$ by writing $e_1 = 2\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix}$, $e_2 = \begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}$ & using linearity of $T$.

⚠ This is not true if $B$ is not a basis (dependencies matter!)

Ex: No linear transf $T: \mathbb{R}^2 \to \mathbb{R}^3$, can map $T(e_1) = \begin{bmatrix}2\\0\\0\end{bmatrix}$, $T(e_2) = \begin{bmatrix}1\\0\\1\end{bmatrix}$ & $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\1\\2\end{bmatrix}$ (value should be $\begin{bmatrix}3\\0\\1\end{bmatrix}$!)

**Proof of Prop**: Since $B$ is a basis for $\mathbb{R}^n$, any vector $v$ in $\mathbb{R}^n$ can be uniquely written as $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$ $\quad \left([v]_B = \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{bmatrix}\right)$

Then apply $T \rightsquigarrow T(v) = T(\alpha_1 v_1 + \cdots + \alpha_n v_n) \underset{T \text{ linear}}{=} T(\alpha_1 v_1) + \cdots + T(\alpha_n v_n)$

$$= \alpha_1 \underbrace{\boxed{T(v_1)}}_{=w_1} + \cdots + \alpha_n \underbrace{\boxed{T(v_n)}}_{=w_n} = \alpha_1 w_1 + \cdots + \alpha_n w_n \quad \text{(prescribed values)} \quad \square$$

- If we want the matrix $A$ giving $T$, need to write $e_1, \ldots, e_n$ as linear comb. of $v_1, \ldots, v_n$ : we need to find $[e_1]_B, \ldots, [e_n]_B$.

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- **Last Time**: 
  - Null Space of $T$: $\mathcal{N}_{(T)} = \{v \text{ in } \mathbb{R}^n : T(v) = 0 \text{ in } \mathbb{R}^m\} = \mathcal{N}(A)$
  - Range of $T$: $\mathcal{R}_{(T)} = \{w \text{ in } \mathbb{R}^m : w = T(v) \text{ for some } v \text{ in } \mathbb{R}^n\} = \mathcal{R}(A)$

**Thm**: Given $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ : $\underbrace{\dim \mathcal{N}(T)}_{\text{nullity}(T)} + \underbrace{\dim \mathcal{R}(T)}_{\text{rank}(T)} = n$.

**Example above**: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\quad A = \begin{bmatrix} 1 & 0 \\ 8 & -4 \\ 1 & 2 \end{bmatrix}$.

- $\text{rank}(T) = \text{rank}(A) = 2$

[Why? $A^T = \begin{bmatrix} 1 & 8 & 1 \\ 0 & -4 & 2 \end{bmatrix} \xrightarrow[R_2 \to R_1 + 2R_2]{} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -\frac{1}{2} \end{bmatrix}$ REF & 2 nonzero rows, so $rk = 2$.]

Range = the plane in $\mathbb{R}^3$ with normal : $\vec{z} = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix} = \begin{vmatrix} i & j & k \\ 1 & 8 & 1 \\ 0 & -4 & 2 \end{vmatrix} = \begin{bmatrix} 20 \\ -2 \\ -4 \end{bmatrix} \rightsquigarrow \mathcal{R}_{(T)} = \{10x - y - 2z$

- nullity $(T)$ : $A = \begin{bmatrix} 1 & 0 \\ 8 & -4 \\ 1 & 2 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_2 \to R_2 - 8R_1}]{} \begin{bmatrix} 1 & 0 \\ 0 & -4 \\ 0 & 2 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2/-4 \\ R_3 \to R_3 - R_2}]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ so solution $\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$ is unique

rank = 2 so rank$(A^T) = 2$ $\overset{\text{Thm}}{=}$ rank$(A)$ $\rightsquigarrow$ nullity $= 0$. $\mathcal{N}(T) = \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$

- Check the theorem : $0 + 2 = 2$ ✓ $(n = 2)$

- Why is the matrix representation useful? Allows for fast compositions!

**Prop** Given $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ l. transf, $G: \mathbb{R}^m \longrightarrow \mathbb{R}^s$ l. transf., the composition $G \circ T: \mathbb{R}^n \longrightarrow \mathbb{R}^s$ given by $G \circ T(\underline{x}) = G(T(\underline{x}))$ is also a linear transformation. If $T(\underline{x}) = A\,\underline{x}$ & $G(\underline{y}) = C\,\underline{y}$ $\quad$ $A$ of size $m \times n$, $C$ — $s \times m$

then $G \circ T(\underline{x}) = (CA)(x)$ $\quad$ ($CA$ of size $s \times n$)

**Example**: 
$T: \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ $\quad T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_3 + x_4 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$G: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\quad G\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} y_1 \\ y_1 + 5y_2 \\ -y_1 + y_2 \end{bmatrix}$ $\quad C = \begin{bmatrix} 1 & 0 \\ 1 & 5 \\ -1 & 1 \end{bmatrix}$ $\rightsquigarrow CA = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 5 & 5 \\ -1 & 1 & 1 & 1 \end{bmatrix}$

Then $G \circ T: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ is $G \circ T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 - x_2 + 5x_3 + 5x_4 \\ -x_1 + x_2 + x_3 + x_4 \end{bmatrix}$