Lecture XXIII: Linear Transformations on Vector Spaces (§5.7)

So far, we've studied • linear transformations $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m \longleftrightarrow$ matrix A of size $m \times n$

• Abstract vector spaces & subspaces of $\mathbb{R}^n$.

TODAY'S GOAL : Combine these 2 notions!

Definition : Fix $\mathbb{V}$, $\mathbb{W}$ two (abstract) vector spaces & $T: \mathbb{V} \longrightarrow \mathbb{W}$, $v \longmapsto T(v)$ in $\mathbb{W}$

We say T is a linear transformation from $\mathbb{V}$ to $\mathbb{W}$ if:

(1) $T(\underbrace{\underline{v} + \underline{u}}_{\text{sum in } \mathbb{V}}) = T(\underline{v}) + T(\underline{u})$ is a vector in $\mathbb{W}$ for all $\underline{u}, \underline{v}$ in $\mathbb{V}$

sum in $\mathbb{W}$

(2) $T(\alpha \cdot \underline{v}) = \alpha \cdot T(\underline{v})$ for all $\underline{v}$ in $\mathbb{V}$, α scalar.

scalar mult in $\mathbb{V}$ — scalar mult in $\mathbb{W}$

§1 Examples

Ex 0 : $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $T(x) = Ax$ (old fashion lin transf on $\mathbb{R}^n$)

Ex 1 : $T: \mathbb{P}_2 \longrightarrow \mathbb{R}$ $T(p(x)) = p(1)$ is a lin transf from $\mathbb{P}_2$ to $\mathbb{R}$

[Eg $T(x^2 + 3x - 2) = 1^2 + 3 - 2 = 2$.] [Call this map = evaluation at 1]

Why? • $T(p(x) + q(x)) = T((p+q)(x)) = (p+q)(1) = p(1) + q(1) = T(p(x)) + T(q(x))$

• $T(\alpha p(x)) = T((\alpha p)(x)) = (\alpha p)(1) = \alpha p(1) = \alpha T(p(x))$.

Generalization : Pick x_0 with $a \leq x_0 \leq b$ $\rightsquigarrow T: C_{[a,b]} \longrightarrow \mathbb{R}$ $T(f(x)) = f(x_0)$

$C_{[a,b]} = \{f: [a,b] \to \mathbb{R} \text{ continuous}\}$

Ex 2 [Taking coordinates with respect to a basis]

Fix a p-dimensional vector space $\mathbb{V}$ with a Basis $\mathcal{B} = \{v_1, \ldots, v_p\}$

Then $T: \mathbb{V} \longrightarrow \mathbb{R}^p$ $T(v) = [v]_{\mathcal{B}}$ coords of v w.r.t. $\mathcal{B}$

is a linear transformation.

Recall: $v = a_1 v_1 + \cdots + a_p v_p$ for $a_1, \ldots, a_p$ scalars (unique because $\mathcal{B}$ is a basis)

Then $[v_{\mathcal{B}}] = [a_1, \ldots, a_p]^T$.

Why? $\, v = a_1 v_1 + \cdots + a_p v_p \rightsquigarrow [v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}$ in $\mathbb{R}^p$

$+\, u = b_1 v_1 + \cdots + b_p v_p \rightsquigarrow [u]_B = \begin{bmatrix} b_1 \\ \vdots \\ b_p \end{bmatrix}$ in $\mathbb{R}^p$

$u+v = (a_1+b_1) v_1 + \cdots + (a_p+b_p) v_p \rightsquigarrow [u+v]_B = \begin{bmatrix} a_1+b_1 \\ \vdots \\ a_p+b_p \end{bmatrix} =$

Similarly $\alpha v = (\alpha a_1) v_1 + \cdots + \alpha a_p v_p \rightsquigarrow [\alpha v]_B = \begin{bmatrix} \alpha a_1 \\ \vdots \\ \alpha a_p \end{bmatrix} = \alpha [v]_B$ in $\mathbb{R}^p$

Remark:
Can combine Ex 1 & 2:

$$\begin{tikzcd} P \text{ in } \mathbb{P}_2 \arrow[r, "{[\ ]_B}"] \arrow[dr, "ev_1"] & \mathbb{R}^3 \arrow[d] \\ & \mathbb{R} \end{tikzcd} \qquad \begin{tikzcd} ax^2+bx+c \arrow[dr, mapsto] & \begin{bmatrix} a \\ b \\ c \end{bmatrix} \arrow[d, "{[1\ 1\ 1]\ 1\times 3 \text{ matrix}}"] \\ & a+b+c \end{tikzcd}$$

$B = \{1, x, x^2\}$

If $P = ax^2+bx+c$

Then $P(1) = a+b+c$ = composition of 2 maps = vertical map & coords in B

We'll use a similar trick to find matrices representing lin transformations whenever the source space $\mathbb{V}$ has FINITE DIMENSION.

Ex 3: Set $T: C[0,1] \longrightarrow \mathbb{R}$ $\quad T(f) = \int_0^1 f(x)\,dx$. is a linear transf

Ex 4: Set $T: C'[0,1] \longrightarrow C[0,1]$ $\quad T(f)(x) = f'(x)$ is a linear transf

$C'[0,1] = \{f$ in $C[0,1]$ with first derivative continuous$\}$ subspace of $C[0,1]$

$\rightsquigarrow$ Can iterate & combine to get differential operators (eg $T(f) = f'' - f' + 100 f$ on $C^2[0,1]$)

- **Q1**: Can we always find l. transf $T: \mathbb{V} \longrightarrow \mathbb{V}$?

  A: YES, the Identity map! $T(v) = v$ for all v in $\mathbb{V}$.

- **Q2**: Can we always find l. transf $T: \mathbb{V} \longrightarrow \mathbb{W}$?

  A YES, the zero map! Write $0_{\mathbb{V}}$ = zero vector in $\mathbb{V}$ $(v + 0_{\mathbb{V}} = v)$

  $0_{\mathbb{W}}$ = ______ $\mathbb{W}$ $(w + 0_{\mathbb{W}} = w)$

Then $T(v) = 0_{\mathbb{W}}$ is a linear transf because $0_{\mathbb{W}} + 0_{\mathbb{W}} = 0_{\mathbb{W}}$
$\alpha \cdot 0_{\mathbb{W}} = 0_{\mathbb{W}}$ for all α.

§2 Basic Properties:

I) **Thm 1**: If $\mathbb{V}$ has a finite basis $B = \{v_1, \ldots, v_p\}$, $T: \mathbb{V} \longrightarrow \mathbb{W}$ linear transformation is completely determined by the vectors $T(v_1), \ldots, T(v_p)$

Note: Don't need finite dimensionality on $\mathbb{W}$

Why? Write $v = a_1 v_1 + \cdots + a_p v_p$

Then $T(v) \underset{\text{by}(1)}{=} T(a_1 v_1) + \cdots + T(a_p v_p) \underset{\text{by}(2)}{=} a_1 \boxed{T(v_1)} + \cdots + a_p \boxed{T(v_p)}$

preassigned values!

Application:

Find a linear Transformation $T: \mathcal{P}_3 \longrightarrow \mathcal{P}_2$ with $T(1) = 2 + x$, $T(x) = x - x^2$, $T(x^2) = 5 - 10x$, $T(x^3) = 2$

Soln: $T(a + bx + cx^2 + dx^3) = a(2+x) + b(x - x^2) + c(5 - 10x) + 2d$
$= (2a + 5c + 2d) + (a + b - 10c)x + (-b)x^2$

Conclusion: T induces a linear transformation $\tilde{T}\ \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$[v]_{B_3} \longmapsto [T(v)]_{B_2}$$

$B_3 = \{1, x, x^2, x^3\}$
$B_2 = \{1, x, x^2\}$

More precisely: $\tilde{T}\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} 2a + 5c + 2d \\ a + b - 10c \\ -b \end{bmatrix} \longleftrightarrow$ matrix $=$

$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \\ 1 & x & x^2 & x^3 \end{array} \\ & \begin{bmatrix} 2 & 0 & 5 & 2 \\ 1 & 1 & -10 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{array}{c} 1 \\ x \\ x^2 \end{array} \end{array}$$

(more on this in §5.9)

(II) Null Space & Range of $T: \mathbb{V} \longrightarrow \mathbb{W}$ l. transf

Definition: $\mathcal{N}(T) = \{ v \text{ in } \mathbb{V} : T(v) = \mathbb{0}_{\mathbb{W}} \}$ (Null space)

$\mathcal{R}(T) = \{ w \text{ in } \mathbb{W} : w = T(v) \text{ for some } v \text{ in } \mathbb{V} \}$ (Range)

Thm 2: (1) $T(\mathbb{0}_{\mathbb{V}}) = \mathbb{0}_{\mathbb{W}}$

(2) $\mathcal{N}(T)$ is a subspace of $\mathbb{V}$

(3) $\mathcal{R}(T)$ is a ______ $\mathbb{W}$

(4) $\mathcal{N}(T) = \mathbb{0}_{\mathbb{V}}$ if and only if T is injective ($T(u) = T(v)$ implies $u = v$)

Note: For $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ (2) and (3) were automatic since they were the Null Space & range of the matrix A representing T. In general, we need to work to prove (2) & (3).

Proof: (1) Write $\mathbb{0}_{\mathbb{V}} = 0 \cdot \mathbb{0}_{\mathbb{V}}$ so $T(\mathbb{0}_{\mathbb{V}}) = T(0 \cdot \mathbb{0}_{\mathbb{V}}) = 0\, T(\mathbb{0}_{\mathbb{V}}) = \mathbb{0}_{\mathbb{W}}$

(2) Need to check (S1), (S2), (S3).

$\mathcal{N}(T)$ (S1) is prop (1) ✓ (S2) If $T(u) = \mathbb{0}_{\mathbb{W}} = T(v)$, then $T(u+v) = \mathbb{0}_{\mathbb{W}} + \mathbb{0}_{\mathbb{W}} = \mathbb{0}_{\mathbb{W}}$

(S3) If $T(u) = 0_V$, then $T(\alpha u) = \alpha T(u) = \alpha\, 0_W = 0_W$

(3) $\underline{R(T)}$ (S1) $0_W = T(0_V)$ by (1), so 0_W belongs to $R(T)$

(S2): $w = T(u)$ in $R(T)$
$\tilde{w} = T(v)$ ______ , then $w + \tilde{w} = T(u+v)$ also in $R(T)$

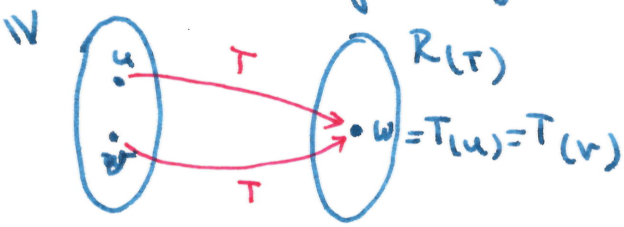
(S3) $w = T(v)$ in $R(T)$ then $\alpha w = \alpha T(u) = T(\alpha u)$ so it's also in $R(T)$

(4) $T(u) = T(v)$ equiv to $0_V = T(v) - T(u) \underset{\substack{\uparrow \\ T \text{ linear}}}{=} T(v-u)$

is equivalent to $v - u$ in $N(T)$.

So $N(T) = 0_V$ is equivalent to having ($T(u) = T(v)$ imply $u = v$)

<u>Remark</u>: To check injectivity at any point w in $R(T)$



implies $u = v$

we <u>only</u> need to check it for $w = 0_W$ ($T(u) = T(v) = 0_W$ means u, v are in $N(T)$)