# Lecture XXIV §5.7 (cont.) Linear Transformations on Vector Spaces
## §5.8 Operations with linear Transformations

**Last Time**: Defined $T: \mathbb{V} \to \mathbb{W}$ linear Transf. (1) $T(u+v) = T(u) + T(v)$
(2) $T(\alpha u) = \alpha T(u)$

- When $\mathbb{V}$ is finite dimension, with basis $B = \{v_1, \ldots, v_p\}$ we can understand $T$ as a map that is linear in the coordinates $[\ ]_B$.
- Defined
  - $\mathcal{N}(T) = \{v \text{ in } \mathbb{V} : T(v) = \mathbb{0}_{\mathbb{W}}\}$ is a subspace of $\mathbb{V}$ (null space)
  - $\mathcal{R}(T) = \{w \text{ in } \mathbb{W} : w = T(v) \text{ for some } v \text{ in } \mathbb{V}\}$ is a subspace of $\mathbb{W}$ (range)

We saw: $T$ is injective if and only if $\mathcal{N}(T) = \{\mathbb{0}_{\mathbb{V}}\}$

This says that checking "$(T(v) = T(\mathbb{0}_{\mathbb{V}})$ implies $v = \mathbb{0}_{\mathbb{V}})$" is enough (& equivalent) to check $T$ injective

**Definition**:
- nullity$(T)$ = dimension of $\mathcal{N}(T)$
- rank$(T)$ = ——— $\mathcal{R}(T)$

**Note**: Both are finite if $\mathbb{V}$ is finite dimensional.

**Thm** Assume $\dim \mathbb{V} = p$, $B = \{v_1, \ldots, v_p\}$ basis for $\mathbb{V}$ & $T: \mathbb{V} \to \mathbb{W}$ l.tr.

Then:
(1) $Sp(T(v_1), \ldots, T(v_p)) = \mathcal{R}(T)$ & so rank $T \leq p$
(2) $T$ is injective if and only if $\{T(v_1), \ldots, T(v_p)\}$ is linearly indep in $\mathbb{W}$.
(3) [Rank-Nullity Thm] $\boxed{\text{rank}(T) + \text{nullity}(T) = \dim \mathbb{V} = p}$

**Proof**: (1) follows from Thm 2 Lecture XXIII: $T$ is determined by the vectors $T(v_1), \ldots, T(v_p)$.
(2) follows from $T$ injective if and only if $\mathcal{N}(T) = \{\mathbb{0}_{\mathbb{V}}\}$.

($\Rightarrow$) $\mathbb{0}_{\mathbb{W}} = a_1 T(v_1) + \cdots + a_p T(v_p) = T\underbrace{(a_1 v_1 + \cdots + a_p v_p)}_{\text{so in } \mathcal{N}(T) \text{ so } = \mathbb{0}_{\mathbb{V}}}$

But $\mathbb{0}_{\mathbb{V}} = a_1 v_1 + \cdots + a_p v_p$ implies $a_1 = \cdots = a_p = 0$ (because $B$ is l.i.)

Conclusion: $\{T(v_1), \ldots, T(v_p)\}$ is l.i.

($\Leftarrow$) Pick $u$ in $\mathcal{N}(T)$ & write it as $u = b_1 v_1 + \cdots + b_p v_p$. Apply $T$

so $\mathbb{0}_{\mathbb{W}} = T(u) = b_1 T(v_1) + \cdots + b_p T(v_p)$ so $b_1 = \cdots = b_p = 0$, so $u = \mathbb{0}_{\mathbb{V}}$.
l.i.

(3) [HARD & FUN!] By (1) notice that rank$(T) \leq p$.
- If rank$(T) = 0$, then $\mathcal{R}(T) = \{\mathbb{0}_{\mathbb{W}}\}$ & so $\mathcal{N}(T) = \mathbb{V}$ & formula holds.
- If rank$(T) = p$, by (1) $\{T(v_1), \ldots, T(v_p)\}$ is a basis for $\mathcal{R}(T)$ so

are l.i & so by (2). T is injective. ~~We have~~: $N(T)=0$.

Again $\text{rank}(T) + \text{nullity}(T) = p + 0 = p$ ✓

• Assume $0 < \text{rank}(T) < p$: $\boxed{\text{Write } r := \text{rank}(T)}$

Since $\text{rank}(T) < p$, we <u>know</u> $\{T(v_1), \ldots, T(v_p)\}$ is not l.i, so T is <u>not</u> injective (by (2)), so $N(T) \neq \{0_V\}$ & we'll be able to pick a basis for $N(T)$ (finite dimensional!) $\boxed{\text{Write } d = \text{nullity of } T}$

• Pick a basis $\{w_1, \ldots, w_r\}$ of $R(T)$ & find $v_1, \ldots, v_r$ in $V$ with $w_1 = T(v_1), \ldots, w_r = T(v_r)$.

• Pick a basis $\{u_1, \ldots, u_d\}$ for $N(T)$.

<u>Claim</u>: $S := \{v_1, \ldots, v_r, u_1, \ldots, u_d\}$ is a basis for $V$, so $\boxed{r+d=p}$ follows.

Pf/(I) <u>Show $S$ is l.i</u>:

Write $a_1 v_1 + \cdots + a_r v_r + \underbrace{b_1 u_1 + \cdots + b_d u_d}_{\downarrow T} = 0_V$ for scalars $a_1 \ldots a_r$, $b_1 \ldots b_d$ (*)

Apply T:

$$\underbrace{a_1 T(v_1) + \cdots + a_r T(v_r)}_{\text{l i}} + \underset{(\text{all } u_i\text{'s in } N(T))}{0_W} = T(0_V) = 0_W. \quad \text{so } a_1 = \cdots = a_r = 0$$

Rewrite (*) $\quad 0_V + b_1 \underbrace{u_1}_{} + \cdots + b_d \underbrace{u_d}_{\text{basis for } N(T)} = 0_V \quad$ so $b_1 = \cdots = b_d = 0$

We conclude $a_1 = \cdots = a_r = b_1 = \cdots = b_d = 0$ so the set is l i.

(II) <u>Show $S$ spans $V$</u>:

Pick any $v$ in $V$, apply T & write $\underset{\text{in } R(T)}{T(v)} = a_1 T(v_1) + \cdots + a_r T(v_r)$ for $a_1, \ldots, a_r$ scalars

$$= T(a_1 v_1 + \cdots + a_r v_r)$$

so $T(v - (a_1 v_1 + \cdots + a_r v_r)) = 0_W$ & so $v - a_1 v_1 - \cdots - a_r v_r$ in $N(T)$

Then, we can find $b_1, \ldots, b_d$ scalars with

$$v - a_1 v_1 - \cdots - a_r v_r = b_1 u_1 + \cdots + b_d u_d$$

Conclusion: $\boxed{v = a_1 v_1 + \cdots + a_r v_r + b_1 u_1 + \cdots + b_d u_d}$

Questions: How to compute a basis for $\mathcal{N}(T)$ & $\mathcal{R}(T)$ when they are not the zero subspace?

A: Work in coordinates when $\dim(W)$ is finite.

Examples: 1 $T: \mathcal{P}_2 \longrightarrow \mathbb{R}$ $T(P(x)) = P(1)$ Use the basis $B = \{1, x, x^2\}$ $\dim \mathcal{P}_2 = 3$

- $\mathcal{N}(T) = \{ax^2 + bx + c \;:\; P(1) = a+b+c = 0 \;\&\; a, b, c \text{ in } \mathbb{R}\}$

$= \{ax^2 + bx + (-a-b)\} = \{a(x^2-1) + b(x-1) \;:\; a, b \text{ in } \mathbb{R}\}$

$= Sp(x^2-1, x-1)$ & they are a basis (l.i.)

$\leadsto$ T is not injective!

- By Thm (1) $\mathcal{R}(T) = Sp(T(1), T(x), T(x^2))$

$T(1) = 1$

$= Sp(1)$ in $\mathbb{R}$ this is just $\mathbb{R}$!

$T(x) = 1$

$T(x^2) = 1$ So Basis $= \{1\}$.

Check: $\operatorname{rank}(T) + \operatorname{nullity}(T) = 1 + 2 = 3 = \dim \mathcal{P}_2$

Example 2: $T: M_{2\times 3} \longrightarrow \mathcal{P}_4$ Basis for $M_{2\times3}$: $B = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$ so $\dim M_{2\times 3} = 6$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \longmapsto (a_{12} + a_{23})x^4 + (2a_{22} + 3a_{13})x^3 + (a_{11} - a_{23})$$

T is a linear transformation because | $B_4 = \{1, x, x^2, x^3, x^4\}$ basis for $\mathcal{P}_4$

$$[T(A)]_{B_4} = \begin{bmatrix} a_{11} - a_{23} \\ 0 \\ 0 \\ 2a_{22} + 3a_{13} \\ a_{12} + a_{23} \end{bmatrix}$$

is a linear map $\tilde{T}: M_{2\times3} \to \mathbb{R}^5$ in the coordinates of matrices with respect to the basis B

- $\mathcal{N}(T) = \left\{A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} : T(A) = 0_{\mathcal{P}_4}\right\} = \left\{A : [T(A)]_{B_4} = \begin{bmatrix} 0\\0\\0\\0\\0 \end{bmatrix}\right\}$

$\hookrightarrow$ coordinates of $0_{\mathcal{P}_4}$ wrt a basis are ALWAYS $0$ in $\mathbb{R}^5$.

$$= \left\{ \begin{array}{l} a_{11} - a_{23} = 0 \\ 2a_{22} + 3a_{13} = 0 \\ a_{12} + a_{23} = 0 \end{array} \right\}$$

$\leadsto$ solve the system in the unknowns $(a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23})$

$$\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & a_{21} & a_{22} & a_{23} \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 3 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \underbrace{0 & \tfrac{2}{3}}_{\text{indep vars.}} & 0 \end{bmatrix}$$

$a_{11} = a_{23}$
$a_{12} = -a_{23}$
$a_{13} = -\frac{2}{3}a_{22}$

so $A = \begin{bmatrix} a_{23} & -a_{23} & -\frac{2}{3}a_{22} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = a_{23}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 & -2/3 \\ 0 & 1 & 0 \end{bmatrix} + a_{21}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$\leadsto$ Basis for $\mathcal{N}(T) = \left\{ \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -2/3 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$, so T is NOT injective!

- $R(T) = Sp\{T(E_{11}), T(E_{12}), T(E_{13}), T(E_{21}), T(E_{22}), T(E_{23})\}$

$$\left.\begin{aligned} T(E_{11}) &= 1 \\ T(E_{12}) &= x^4 \\ T(E_{13}) &= 3x^3 \end{aligned}\right\} \text{only need these!}$$

$$T(E_{21}) = 0$$

$$T(E_{22}) = 2x^3$$

$$T(E_{23}) = x^4 - 1$$

(know these 6 vectors are not l.i because $T$ is not injective!)

$\Rightarrow R(T) = Sp(1, x^4, x^3)$

& Basis $= \{1, x^3, x^4\}$

$\underset{3}{\underset{\shortparallel}{\text{rank}(T)}} + \underset{3}{\underset{\shortparallel}{\text{nullity}(T)}} = 6$ ✓

## §5.8 Operations with Linear Transformations

Our next goal: describe operations between linear transf (in analogy with operations $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m \longleftrightarrow$ operations on matrices)

| Oper | Matrices | $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ | $T: V \longrightarrow W$ |
|---|---|---|---|
| (I) | Addition in $M_{m\times n}$ <br> $(A+C)_{ij} = A_{ij} + C_{ij}$ | Addition <br> $(T_1+T_2)(v) = T_1(v) + T_2(v)$ | Addition $T_1+T_2$ <br> $(T_1+T_2)(v) = T_1(v) + T_2(v)$ |
| (II) | Scalar mult <br> $(\lambda\cdot A)_{ij} = \lambda A_{ij}$ | scalar multiplication <br> $(\lambda T)(v) = \lambda T(v)$ | scalar multiplication <br> $(\lambda T)(v) = \lambda T(v)$ |
| (III) | MATRIX MULTIPLICATION <br> $CA$ in $M_{s\times n}$ <br> for $A$ in $M_{m\times n}$ <br> $C$ in $M_{s\times m}$ | COMPOSITION <br> $T_1: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ <br> $T_2: \mathbb{R}^m \longrightarrow \mathbb{R}^s$ <br> $T_2\circ T_1 : \underbrace{\mathbb{R}^n \xrightarrow{T_1} \mathbb{R}^m \xrightarrow{T_2} \mathbb{R}^s}_{T_2\circ T_1}$ | COMPOSITION <br> $T_1: V \longrightarrow W$ <br> $T_2: W \longrightarrow U$ <br> $T_2\circ T_1 : \underbrace{V \to W \to U}_{T_2\circ T_1}$ <br> $T_2\circ T_1(v) = T_2(\underbrace{T_1(v)}_{\text{in } W})$ in $U$ |

$T_1$ has matrix $A$
$T_2$ has matrix $C$

**Prop**: Operations (I) & (II) turn $\left\{ \underset{T \text{ linear transf}}{T: V \longrightarrow W} \right\}$ for fixed $V, W$ vector spaces into an abstract vector space!

Let's do some examples next time.