Lecture XXV: §5.8 (cont.) Operations with linear Transformations

Recall: (1) $T_1, T_2 : \mathbb{V} \longrightarrow \mathbb{W}$ linear Transfs, we can define

$+$: $T = T_1 + T_2 : \mathbb{V} \longrightarrow \mathbb{W}$ $\quad T(v) = \underbrace{T_1(v) + T_2(v)}_{\text{vectors in } \mathbb{W}}$ l. transf

$\alpha\cdot$: $\alpha \cdot T_1 : \mathbb{V} \longrightarrow \mathbb{W}$ $\quad (\alpha\cdot T)(v) = \alpha\, T(v)$ again a l. transf

These 2 operations turn the set $\{T : \mathbb{V} \to \mathbb{W}\}$ into a vector space!

(2) If $T_1 : \mathbb{V} \longrightarrow \mathbb{W}$ l.t., $T_2 : \mathbb{W} \longrightarrow \mathbb{U}$ l.t., we define the composition $T_2 \circ T_1 : \mathbb{V} \xrightarrow{T_1} \mathbb{W} \xrightarrow{T_2} \mathbb{U}$

as the linear transformation $T_2 \circ T_1(v) = T_2(\underbrace{T_1(v)}_{\text{in } \mathbb{W}})$ for all v in $\mathbb{V}$.

Note: If $\mathbb{V} \neq \mathbb{U}$, we don't have $T_1 \circ T_2$! (Order matters!)

III Examples & properties

Ex 1. $T_1 : \mathcal{P}_2 \longrightarrow \mathbb{R}$, $P \longmapsto P(1)$, $\quad T_2 : \mathcal{P}_2 \longrightarrow \mathbb{R}$, $P \longmapsto P'(2)$ $\quad$ both are linear Transform.

Why? Set $\mathcal{B} = \{x^2, x, x^2\}$ our preferred basis for $\mathcal{P}_2$.

Then $P(x) = ax^2 + bx + c \rightsquigarrow T_1(P) = a + b + c$

$T_2(P) = (2ax + b)_{(2)} = 4a + b$

T_1, T_2 are linear in $[P(x)]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ so they are linear transf!

(1) New l. transf $T = T_1 + T_2 : \mathcal{P}_2 \longrightarrow \mathbb{R}$

$P \longmapsto P(1) + P'(2)$

In Terms of a, b, c: $(T_1 + T_2)(P) = a + b + c + 4a + b = 5a + 2b + c$.

- $\mathcal{N}(T) = \{5a + 2b + c = 0\} = \{ax^2 + bx + (-5a - 2b) : a, b \text{ in } \mathbb{R}\}$
$= \{a\underbrace{(x^2 - 5)} + b\underbrace{(x - 2)} : a, b \text{ in } \mathbb{R}\}$

So a basis for $\mathcal{N}(T)$ is $\{x^2 - 5, x - 2\}$. vectors in $\mathcal{P}_2$

- **Note** $\mathcal{N}(T_1)$ has basis $\{x^2 - 1, x - 1\}$ (last time),

$\mathcal{N}(T_2) = \{ax^2 + (-4a)x + c\} = \{a\underbrace{(x^2 - 4x)} + c\cdot\underbrace{1} : a, c \text{ in } \mathbb{R}\}$

so has basis $\{x^2 - 4x, 1\}$ $\quad$ vectors in $\mathcal{P}_2$

$\mathcal{N}(T_1)$ & $\mathcal{N}(T_2)$ are NOT related to $\mathcal{N}(T_1 + T_2)$.

- $\mathcal{R}(T)$ has dimension $= \dim \mathcal{P}_2 - \text{nullity}(T) = 3 - 2 = 1$

$\mathcal{R}(T)$ is a subspace of $\mathbb{R}$, which has $\dim \mathbb{R} = 1$, we conclude $\boxed{\mathcal{R}(T) = \mathbb{R}}$

(2) New l. transf : $3T_1 : \mathcal{P}_2 \longrightarrow \mathbb{R}$ $(3T_1)_p = 3P_{(1)} = 3(a+b+c)$

$\mathcal{N}(3T_1) = \mathcal{N}(T_1)$, $\mathcal{R}(3T_1) = \mathcal{R}(T_1) = \mathbb{R}$

Prop : • If $\lambda \neq 0$ $\mathcal{N}(\lambda T) = \mathcal{N}(T)$ & $\mathcal{R}(\lambda T) = \mathcal{R}(T)$

• If $\lambda = 0$ $\mathcal{N}(\lambda T) = \mathbb{V}$ because $0.T : \mathbb{V} \longrightarrow \mathbb{W}$ is the zero map $(0T)_{(v)} = 0_{\mathbb{W}}$ for all v in $\mathbb{V}$.

$$\mathcal{R}(\lambda T) = \{0_{\mathbb{W}}\}$$

Ex 2 : $T_1 : M_{2\times3} \longrightarrow \mathcal{P}_3$, $T_2 : \mathcal{P}_3 \longrightarrow \mathbb{R}^2$ What is $T_2 \circ T_1$?

$A \longmapsto a_{11}x^3 + (a_{12}-a_{13})x^2 + a_{23}$ $\underline{a}x^3 + \underline{b}x^2 + \underline{c}x + \underline{d} \longmapsto \begin{bmatrix} a-b \\ a-c+d \end{bmatrix}$

Composition $T = T_2 \circ T_1$: $M_{2\times3} \xrightarrow{T_1} \mathcal{P}_3 \xrightarrow{T_2} \mathbb{R}^2$

$$A \longmapsto \underbrace{a_{11}}_{=a} x^3 + \underbrace{(a_{12}-a_{13})}_{b} x^2 + \underbrace{a_{23}}_{d} \longmapsto \begin{bmatrix} a_{11} - (a_{12}-a_{13}) \\ a_{11} - 0 + a_{23} \end{bmatrix}$$

$c = 0$

In conclusion: ; $T : \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \longmapsto \begin{bmatrix} a_{11} - a_{12} + a_{13} \\ a_{11} + a_{23} \end{bmatrix}$

In general: $T_1 : \mathbb{V} \longrightarrow \mathbb{W}$, $T_2 : \mathbb{W} \longrightarrow \mathbb{U}$

Q1 Are $\mathcal{N}(T_1)$ & $\mathcal{N}(T_2 \circ T_1)$ related? (Both are subspaces of $\mathbb{V}$)

Yes! If $T_1(v) = 0_{\mathbb{W}}$, then $T_2(T_1(v)) = T_2(0_{\mathbb{W}}) = 0_{\mathbb{U}}$ (T_2 is linear because)

So $\mathcal{N}(T_1)$ lies in $\mathcal{N}(T_2 \circ T_1)$ (Symbol : $\boxed{\mathcal{N}(T_1) \subseteq \mathcal{N}(T_2 \circ T_1)}$)

Q2. Are $\mathcal{R}(T_2)$ & $\mathcal{R}(T_2 \circ T_1)$ related? Both are subspaces of $\mathbb{U}$)

Yes! If u in $\mathcal{R}(T_2 \circ T_1)$, then $u = T_2 \circ T_1(v) = T_2(\underbrace{T_1(v)}_{\text{in } \mathbb{W}})$ for some v in $\mathbb{V}$

so u in $\mathcal{R}(T_2)$ by definition.

Conclusion: $\boxed{\mathcal{R}(T_2 \circ T_1) \subseteq \mathcal{R}(T_2)}$.

Example above: $\mathcal{N}(T_1) = \left\{ \begin{pmatrix} 0 & a_{12} & a_{12} \\ a_{21} & a_{22} & 0 \end{pmatrix} \right\}$ so basis $\{E_{12} + E_{13}, E_{21}, E_{22}\}$

$\mathcal{N}(T_2 \circ T_1) = \left\{ \begin{pmatrix} -a_{23} & a_{13}-a_{23} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \right\}$ so basis $\{E_{23} - E_{12} - E_{11}, E_{12} + E_{13}, E_{21}, E_{22}\}$

• $\operatorname{rank}(T_2 \circ T_1) = 6 - 4 = 2$ & lies in $\mathbb{R}^2$, so $\mathcal{R}(T_2 \circ T_1) = \mathbb{R}^2$

• $\mathcal{R}(T_2) = \operatorname{Sp}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \mathbb{R}^2$

§2. Surjective or Onto Transformations

Def: A linear transformation $T: \mathbb{V} \to \mathbb{W}$ is ONTO (or surjective) if $R(T) = \mathbb{W}$

Note: If $\mathbb{W}$ has finite dimension, it's enough to check $\operatorname{rank}(T) = \dim \mathbb{W}$

$\operatorname{rank}(T) = \dim R(T)$.

Example 1: $T: \mathcal{P}_3 \to \mathcal{P}_2$ $\quad T(P(x)) = P'(x)$. is ONTO.

If $P = ax^3 + bx^2 + cx + d$ $\quad T(P) = 3a\,x^2 + 2b\,x + c \cdot 1$

$$R(T) = \operatorname{Sp}\{T(1), T(x), T(x^2), T(x^3)\}$$
$$= \operatorname{Sp}\{0, 1, 2x, 3x^2\} = \operatorname{Sp}\{1, x, x^2\} = \mathcal{P}_2$$

Example 2 $\tilde{T}: \mathcal{P}_3 \to \mathcal{P}_3$ $\quad \tilde{T}(P(x)) = P'(x)$ is NOT onto because $R(\tilde{T}) = \mathcal{P}_2$, not $\mathcal{P}_3$!

§3 Invertible Transformations: Analogs to invertible MATRICES

Def $T: \mathbb{V} \to \mathbb{W}$ l. transformation is invertible if we can find $S: \mathbb{W} \to \mathbb{V}$ linear transformation with $S \circ T: \mathbb{V} \to \mathbb{V}$ $\quad S \circ T = Id_{\mathbb{V}}$ $\quad$ ($S \circ T(v) = v$ for all v in $\mathbb{V}$)

& $T \circ S: \mathbb{W} \to \mathbb{W}$ $\quad T \circ S = Id_{\mathbb{W}}$

Write $S = T^{-1}$, call T isomorphism

Remark If $\mathbb{V} = \mathbb{W} = \mathbb{R}^n$, then T is invertible if and only if the matrix A representing T is invertible (because the matrix for S will be A^{-1})

Proposition: If T is invertible, $T: \mathbb{V} \to \mathbb{W}$, then 1) $N(T) = \{0_{\mathbb{V}}\}$ (T injective)

2) $R(T) = \mathbb{W}$ (T onto)

Proof: 1) $N(T) \subseteq N(\underbrace{T^{-1} \circ T}_{= Id_{\mathbb{V}}}) = \{0_{\mathbb{V}}\}$ so $N(T) = \{0_{\mathbb{V}}\}$

2) $R(\underbrace{T \circ T^{-1}}_{Id_{\mathbb{W}}}) = \mathbb{W} \subseteq R(T)$ so $R(T) = \mathbb{W}$

THM 1: If T is injective & onto, then T is invertible (T^{-1} exists & it is a linear transf.)

THM 2 If $\dim \mathbb{V} = \dim \mathbb{W} = p$, then we have $N(T) = \{0_{\mathbb{V}}\}$ if and only if $R(T) = \mathbb{W}$. (So we to only check one of the 2 conditions)

Pf/ Use rank-nullity theorem. □

<u>Example</u>: $\dim \mathbb{W} = p$ $\quad T: \mathbb{V} \longrightarrow \mathbb{R}^p \quad T(v) = [v]_B$ is invertible.

$B = \{v_1, \ldots, v_p\}$ basis for $\mathbb{W}$

$N(T) = \{0_{\mathbb{W}}\} \qquad [v] = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ implies $v = 0_{\mathbb{W}} = 0v_1 + \cdots + 0v_p$.

<u>Inverse</u>. $T^{-1}: \mathbb{R}^p \longmapsto \mathbb{V} \qquad T^{-1}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}\right) = a_1 v_1 + \cdots + a_p v_p$

<u>Thm</u>: Given $\mathbb{V}$, $\mathbb{W}$ of the <u>same dimension</u> p, we can always find an <u>invertible</u> linear transformation $T: \mathbb{V} \longrightarrow \mathbb{W}$.

Pf/

$$\begin{tikzcd} \mathbb{V} \arrow[r, "{[\ ]_B}", "T_1"'] & \mathbb{R}^n \arrow[d, "Id"] \\ \mathbb{W} \arrow[r, "T_2", "{[\ ]_{B'}}"'] & \mathbb{R}^n \end{tikzcd}$$

Write $T = T_2^{-1} \circ T_1$ is invertible because it is composition of invertibles.