Lecture XXVI: §5.9 Matrix representations for linear transformations

GOAL: Find matrices to represent l.t $T: \mathbb{V} \longrightarrow \mathbb{W}$ when $\mathbb{V}$ & $\mathbb{W}$ are finite dimensional

Last time: If $\dim \mathbb{V} = p$, with basis B for $\mathbb{V}$, then $T_B: \mathbb{V} \longrightarrow \mathbb{R}^p$, $v \longmapsto [v]_B$ is an invertible l.t.

Why? $S: \mathbb{R}^p \longrightarrow \mathbb{V}$, $\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} \longmapsto v = a_1 v_1 + \cdots + a_p v_p$ if $B = \{v_1, \ldots, v_p\}$ is the inverse (linear) map.

Rem: Essentially, we can restrict to $\tilde{T}: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear transf (if $\dim \mathbb{V}$ is finite)

Why? If $T: \mathbb{V} \longrightarrow \mathbb{W}$ and $\dim \mathbb{V} = p$, we know $R(T)$ has dim at most p (spanned by $\{T(v_1), \ldots, T(v_p)\}$ where $B = \{v_1, \ldots, v_p\}$ basis for $\mathbb{V}$.

STEP 1: Replace $\mathbb{W}$ by $R(T)$, so we can assume BOTH $\mathbb{V}$ & $\mathbb{W}$ are finite dimensional

STEP 2:

$$\begin{array}{ccc} \mathbb{V} & \xrightarrow{T} & \mathbb{W} \\ {\scriptstyle \text{invertible}}\ \downarrow T_B & & T_C \downarrow\ {\scriptstyle \text{invertible.}} \\ \mathbb{R}^p & \xrightarrow{\tilde{T}} & \mathbb{R}^q \end{array}$$

Pick B basis for $\mathbb{V}$, $\dim \mathbb{V} = p$

C " " $\mathbb{W}$, $\dim \mathbb{W} = q$

$$\tilde{T} = T_C \circ T \circ T_B^{-1}.$$

$$\tilde{T}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}\right) = \left[T(a_1 v_1 + \cdots + a_p v_p)\right]_C$$

- $\tilde{T}$ gives us a way of representing T via a $(q \times p)$ matrix.
- We call the matrix of $\tilde{T}$ the matrix of T relative to the basis B & C for $\mathbb{V}$ & $\mathbb{W}$ respectively

Q How to find this matrix?

$T: \mathbb{V} \longrightarrow \mathbb{W}$ linear transf

$B = \{v_1, \ldots, v_p\}$ basis for $\mathbb{V}$,

$C = \{w_1, \ldots, w_q\}$ ——— $\mathbb{W}$.

Write
$$\begin{cases} T(v_1) = a_{11} w_1 + \cdots + a_{q1} w_q \\ T(v_2) = a_{12} w_1 + \cdots + a_{q2} w_q \\ \vdots \\ T(v_p) = a_{1p} w_1 + \cdots + a_{qp} w_q \end{cases}$$

$\rightsquigarrow (a_{ij})$ is a $q \times p$ matrix.

i: index of w's; j: index of v's

$$\text{Matrix } [T]_{BC} = \begin{array}{c} \begin{array}{ccc} T(v_1) & & T(v_p) \end{array} \\ \begin{bmatrix} a_{11} & \cdots & a_{p1} \\ \vdots & & \\ a_{q1} & \cdots & a_{qp} \end{bmatrix} \end{array} \begin{array}{c} w_1 \\ \vdots \\ w_q \end{array} = \begin{array}{c} \begin{array}{ccc} v_1 & & v_p \end{array} \\ \begin{bmatrix} [T(v_1)]_C & \cdots & [T(v_p)]_C \end{bmatrix} \end{array} \begin{array}{c} w_1 \\ \vdots \\ w_q \end{array}$$

Q What is this matrix good for?

Representation Theorem: The matrix $[T]_{BC}$ is the unique matrix A of size $q \times p$ satisfying $A\,[v]_B = [T(v)]_C$.

$$\Big(\underbrace{\underset{\substack{\downarrow \\ q\times p}}{[T]_{BC}}\ \underset{\substack{\downarrow \\ \text{in } \mathbb{R}^p \ (p\times 1 \text{ matrix})}}{[v]_B}}_{q\times 1 \text{ matrix}} = [T(v)]_C \Big)$$

§1 Examples:

Ex 1. $T: \mathcal{P}_2 \longrightarrow \mathbb{R}^2$ Find $[T]_{BC}$ $B = \{\underset{v_1}{x^2}, \underset{v_2}{x}, \underset{v_3}{1}\}$

$p(x) = ax^2+bx+c \longmapsto \begin{bmatrix} a-b \\ 2a+c \end{bmatrix}$ $C = \{\underset{w_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \underset{w_2}{\begin{bmatrix}0\\1\end{bmatrix}}\}$

$T(x^2) = \begin{bmatrix}1\\2\end{bmatrix} = e_1 + 2e_2$ $(a=1,\ b=c=0)$ $\rightsquigarrow [T(x^2)]_C = \begin{bmatrix}1\\2\end{bmatrix}$

$T(x) = \begin{bmatrix}-1\\0\end{bmatrix}$ $(a=c=0 \ \ b=1)$

$T(1) = \begin{bmatrix}0\\1\end{bmatrix}$ $(a=b=0 \ \ c=1)$

$$[T]_{BC} = \begin{array}{c} \begin{array}{ccc} T(v_1) & T(v_2) & T(v_3) \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{array}{c} w_1 \\ w_2 \end{array} \end{array}$$

Ex 1.1: Same T, find $[T]_{BC'}$ with $C' = \{\underset{w_1'}{\begin{bmatrix}1\\1\end{bmatrix}}, \underset{w_2'}{\begin{bmatrix}-1\\0\end{bmatrix}}\}$

$T(x^2) = \begin{bmatrix}1\\2\end{bmatrix} = 2 w_1' + 1 w_2'$ (*) $\rightsquigarrow [T(x^2)]_{C'} = \begin{bmatrix}2\\1\end{bmatrix}$

$T(x) = \begin{bmatrix}-1\\0\end{bmatrix} = 0\cdot w_1' + 1 w_2' \rightsquigarrow [T(x)]_{C'} = \begin{bmatrix}0\\1\end{bmatrix}$ $\rightsquigarrow [T]_{BC'} = \begin{array}{c} \begin{array}{ccc} T(v_1) & T(v_2) & T(v_3) \end{array} \\ \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{array}{c} w_1' \\ w_2' \end{array} \end{array}$

$T(1) = \begin{bmatrix}0\\1\end{bmatrix} = w_1' + w_2' \rightsquigarrow [T(1)]_{C'} = \begin{bmatrix}1\\1\end{bmatrix}$

- Check Rep. Thm: $[p]_B = \begin{bmatrix}a\\b\\c\end{bmatrix}$ for both C & C'.

- $[T(p)]_C = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix} a-b \\ 2a+c \end{bmatrix}$ ✓

- $[T(p)]_{C'} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix} 2a+c \\ a+b+c \end{bmatrix}$

and $T(p) = (2a+c)\underset{\begin{bmatrix}1\\1\end{bmatrix}}{w_1'} + (a+b+c)\underset{\begin{bmatrix}-1\\0\end{bmatrix}}{w_2'} = \begin{bmatrix} 2a+c-a-b \\ 2a+c+0 \end{bmatrix} = \begin{bmatrix} a-b \\ 2a+c \end{bmatrix}$ ✓

Why (*)? Write $\begin{bmatrix}1\\2\end{bmatrix} = a\, w_1' + b\, w_2' = a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix} a-b \\ a \end{bmatrix}$

Identify coefficients: $\begin{array}{l} 1 = a-b \\ 2 = a \end{array}$ so $a=2$ & $b=1$. Then $\begin{bmatrix}1\\2\end{bmatrix} = 2w_1' + w_2'$

Ex 2 Given $T: M_{2\times 2} \longrightarrow \mathcal{P}_3$ find $[T]_{BC}$, $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto (a+d)x^2 + ax + (b-3c) \qquad C = \{x^2, x, 1\}$$

$$T(E_{11}) = T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \underset{\substack{a=1 \\ b=c=d=0}}{=} x^2 + x \rightsquigarrow [T(E_{11})]_C = \begin{array}{c} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \end{array}\begin{array}{c} x^2 \\ x \\ 1 \end{array}$$

$$T(E_{12}) = T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \underset{\substack{a=c=d=0 \\ b=1}}{=} 0\cdot x + 0 \cdot x + 1 \rightsquigarrow [T(E_{12})]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T(E_{21}) = T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \underset{\substack{c=1 \\ a=b=d=0}}{=} 0\cdot x + 0 \cdot x - 3 \rightsquigarrow [T(E_{21})]_C = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$$

$$T(E_{22}) = T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \underset{\substack{d=1 \\ a=b=c=0}}{=} 1x + 0 \cdot x + 0 \rightsquigarrow [T(E_{22})]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightsquigarrow [T]_{BC} = \begin{array}{c} \begin{array}{cccc} T(E_{11}) & T(E_{12}) & T(E_{21}) & T(E_{22}) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ x^2 \\ x \\ 1 \end{array}$$

Ex 3 Given $T: \mathbb{R}^p \longrightarrow \mathbb{R}^q$ $\quad T(x) = Ax$ $\quad$ Then $A = [T]_{BC}$

A of size $q \times p$.

where $B = \{e_1, \dots, e_p\}$ in $\mathbb{R}^p$

$C = \{e_1, \dots, e_q\}$ in $\mathbb{R}^q$

Why? $A = [T(e_1), \dots, T(e_p)]$ & $[w]_C = w$ for every w in $\mathbb{R}^q$.

§2 Algebraic Properties:

· Question: given $+$, $\alpha\cdot$ & composition of l. transf., how to relate the matrices?

THM: Fix 2 finite dimensional vector space $\mathbb{V}$ & $\mathbb{W}$ with $\dim \mathbb{V} = p$, $\dim \mathbb{W} = q$, with bases B & C, respectively. Consider 2 linear transformations:

$$T_1: \mathbb{V} \longrightarrow \mathbb{W} \quad \& \quad T_2: \mathbb{V} \longrightarrow \mathbb{W}$$

Then: (1) The matrix $[T_1 + T_2]_{BC} = [T_1]_{BC} + [T_2]_{BC}$

(2) The matrix $[\alpha T_1]_{BC} = \alpha [T_1]_{BC}$ for any scalar α.

Example (last time) $T_1: \mathcal{P}_2 \longrightarrow \mathbb{R}$, $p \longmapsto p(1)$, $(ax^2+bx+c) \longmapsto a+b+c$; $T_2: \mathcal{P}_2 \longrightarrow \mathbb{R}$, $p \longmapsto p'(2)$, $ax^2+bx+c \longmapsto 4a+b$ $\quad B = \{x^2, x, 1\}$, $C = \{1\}$

$[T_1]_{BC} = [1 \ \ 1 \ \ 1]$

$[T_2]_{BC} = [4 \ \ 1 \ \ 0]$ & $[T_1 + T_2]_{BC} = [5 \ \ 2 \ \ 1]$, $[3T_1]_{BC} = [3 \ \ 3 \ \ 3]$