Lecture XXVII: §5.9 Matrix representations & §6.2 Determinants

Recall: Given 2 vector spaces $\mathbb{V}$, $\mathbb{W}$, $\dim \mathbb{V} = p$, $\dim \mathbb{W} = q$ and 2 bases $B = \{v_1, \ldots, v_p\}$ for $\mathbb{V}$, $C = \{w_1, \ldots, w_q\}$ for $\mathbb{W}$, and a linear transformation $T: \mathbb{V} \to \mathbb{W}$

we can construct a $(q \times p)$ matrix $[T]_{BC} = \Big[[T(v_1)]_C, \ldots [T(v_p)]_C \Big]$

- Defining property: $\underbrace{[T(v)]_C}_{q\times 1} = \underbrace{[T]_{BC}}_{q\times p} \underbrace{[v]_B}_{p\times 1}$ for any v in $\mathbb{V}$
- Compatible with $+$ & scalar multiplications

THM: Given $T_1: \mathbb{V} \longrightarrow \mathbb{W}$, $T_2: \mathbb{W} \longrightarrow \mathbb{U}$ 2 linear transformations $\dim \mathbb{V} = p$, $\dim \mathbb{W} = q$, $\dim \mathbb{U} = r$, basis B for $\mathbb{V}$, C for $\mathbb{W}$, D for $\mathbb{U}$, then:

$$[\underbrace{T_2 \circ T_1}_{r\times p}]_{BD} = \underset{\substack{\downarrow \\ r\times q}}{[T_2]_{CD}} \; \underset{\substack{\downarrow \\ q\times p}}{[T_1]_{BC}}$$

sizes: $r\times p$ $\quad r\times q \quad q\times p$

Why?

$$[T_2\circ T_1(v)]_D = [T_2(T_1(v))]_D \overset{\overset{\text{defining prop for } T_1}{\uparrow}}{=} [T_2]_{CD}\,[T_1(v)]_C$$

$$\shortparallel$$

$$\underbrace{[T_2\circ T_1]_{BD}}\,[v]_B \qquad = \underbrace{[T_2]_{CD}\,[T_1]_{BC}}\,[v]_B \quad \text{for all } v \text{ in } \mathbb{V}$$

these matrices must agree.

Example: $T_1: \mathbb{M}_{2\times 2} \longrightarrow \mathcal{P}_2$ $\qquad B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto (a+d)x^2 + ax + (b-3c)$ $\qquad C = \{x^2, x, 1\}$

$T_2: \mathcal{P}_2 \longrightarrow \mathbb{R}^2$ $\quad T_2(p) = \begin{bmatrix} p(1) \\ p'(1) \end{bmatrix}$ $\qquad D = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

$ax^2 + bx + c \longmapsto \begin{bmatrix} a+b+c \\ 2a+b \end{bmatrix}$

$$(T_2\circ T_1)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T_2\big((a+d)x^2 + ax + (b-3c)\big) = \begin{bmatrix} (a+d) + a + b - 3c \\ 2(a+d) + a \end{bmatrix} = \begin{bmatrix} 2a+b-3c+d \\ 3a+d \end{bmatrix}$$

So
$$[T_2\circ T_1]_{BD} = \begin{array}{c} \\ \\ \end{array}\left[\begin{array}{cccc} T(E_{11}) & T(E_{12}) & T(E_{21}) & T(E_{22}) \\ 2 & 1 & -3 & 1 \\ 3 & 0 & 0 & 1 \end{array}\right]\begin{array}{c} \\ e_1 \\ e_2 \end{array} \quad (2\times 4) \text{ matrix}$$

Also: $[T_1]_{BC} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{bmatrix}$ (last time)

$[T_2]_{CD} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

& $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -3 & 1 \\ 3 & 0 & 0 & 1 \end{bmatrix}$

§2 Determinants

Definition: Let $A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a 2×2 matrix. The **determinant** of A is given by $\det A = a_{11} a_{22} - a_{12} a_{21}$.

Notation:
$$\det A = \begin{array}{cc} + & - \\ \end{array} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Examples: $\det A = ?$ for $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

Solution:
$$\det A = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 1 \cdot 3 - 2 \cdot (-1) = 3 + 2 = 5$$

$$\det B = \begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix} = 3 \cdot 8 - 6 \cdot 4 = 24 - 24 = 0$$

Q: How to extend this to bigger square matrices? Use an "inductive procedure": define $\det(A)$ for A a $n \times n$ matrix in terms of submatrices of A with 1 row & 1 column less (similar to the way we defined cross-products).

Definition Let $A = (a_{ij})$ be an $n \times n$ matrix and 2 numbers r, s with $1 \leq r, s \leq n$, we let M_{rs} be the $(n-1) \times (n-1)$ matrix obtained by deleting the r^{th} row & the s^{th} column. The new matrix is called the (r,s)-minor of A. The numbers

$$A_{ij} = (-1)^{i+j} \det M_{ij}$$

are called the cofactors of A (or signed minors)

Using the cofactors, we define the determinant of A

Def
$$\det A := a_{\underline{11}} A_{11} + a_{\underline{12}} A_{12} + \cdots + a_{\underline{1n}} A_{1n} = \sum_{j=1}^{n} a_{1j} A_{1j}$$

Idea:
$$\rightarrow \begin{vmatrix} \boxed{a_{11} \quad a_{12} \quad \cdots \quad a_{1n}} \\ a_{21} \\ \vdots \\ a_{n1} \quad \cdots \quad a_{nn} \end{vmatrix} = a_{11} (-1)^{1+1} \overset{\text{remove row 1 \& col 1}}{\begin{vmatrix} a_{22} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}} + a_{12} (-1)^{1+2} \overset{\text{remove row 1, col 2}}{\begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix}} + \cdots$$

Example 1

$$\det\begin{bmatrix}3&2&1\\2&1&-3\\4&0&1\end{bmatrix} = 3\cdot(-1)^{1+1}\begin{bmatrix}3&2&1\\2&1&-3\\4&0&1\end{bmatrix} + 2(-1)^{1+2}\begin{vmatrix}3&2&1\\2&1&-3\\4&0&1\end{vmatrix}$$

$$+ 1(-1)^{1+3}\begin{vmatrix}3&2&1\\2&1&-3\\4&0&1\end{vmatrix}$$

$$= 3\,(1\cdot1-0)\ -2\ (2\cdot1-4(-3)) + 1\,(2\cdot0-1\cdot4)$$

$$= 3\cdot1\ -\ 2\,\underbrace{(2+12)}_{14} + (-4) = 3-28-4 = -29$$

Example 2: $A = \begin{bmatrix}1&2&0&2\\-1&2&3&1\\-3&2&-1&0\\2&-3&-2&1\end{bmatrix}$ Show that $\det A = -63$

$$\det A = 1(-1)^{1+1}\begin{vmatrix}1&2&0&2\\-1&2&3&1\\-3&2&-1&0\\2&-3&-2&1\end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix}1&2&0&2\\-1&2&3&1\\-3&2&-1&0\\2&-3&-2&1\end{vmatrix} + 0 + 2\cdot(-1)^{1+4}\begin{vmatrix}1&2&0&2\\-1&2&3&1\\-3&2&-1&0\\2&-3&-2&1\end{vmatrix}$$

- $\det\begin{bmatrix}2&3&1\\2&-1&0\\-3&-2&1\end{bmatrix} = 2(-1)^{1+1}\begin{vmatrix}2&3&1\\2&-1&0\\-3&-2&1\end{vmatrix} + 3(-1)^{1+2}\begin{vmatrix}2&3&1\\2&-1&0\\-3&-2&1\end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix}2&3&1\\2&-1&0\\-3&-2&1\end{vmatrix}$

$$= 2\ (-1-0)\ -3\ (2-0)\ +\ (2(-2)-(-1)(-3))$$

$$= (-2)\ -6 + (-4-3) = -8-7 = -15$$

$$\det\begin{bmatrix}-1&3&1\\-3&-1&0\\2&-2&1\end{bmatrix} = (-1)(-1)^{1+1}\begin{vmatrix}-1&3&1\\-3&-1&0\\2&-2&1\end{vmatrix} + 3(-1)^{1+2}\begin{vmatrix}-1&3&1\\-3&-1&0\\2&-2&1\end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix}-1&3&1\\-3&-1&0\\2&-2&1\end{vmatrix}$$

$$= -1\cdot(-1-0)\ -3\ (-3-0)\ +\ (-3(-2)-(-1)\cdot2)$$

$$= 1 + 9 + (6+2) = 1+9+8 = +18$$

$$\det\begin{bmatrix}-1&2&3\\-3&2&-1\\2&-3&-2\end{bmatrix} = (-1)(-1)^{1+1}\begin{vmatrix}-1&2&3\\-3&2&-1\\2&-3&-2\end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix}-1&2&3\\-3&2&-1\\2&-3&-2\end{vmatrix} + 3(-1)^{1+3}\begin{vmatrix}-1&2&3\\-3&2&-1\\2&-3&-2\end{vmatrix}$$

$$= -1\,(2(-2)-(-1)(-3))\ -2\ (-3(-2)-2(-1)) + 3\,(-3(-3)-2\cdot2)$$

$$= -1\,(-4-3)\ -2\,(6+2) + 3\,(9-4)$$

$$= 7-16+15 = 6$$

Conclusion: $\det A = 1\,(-15) - 2\cdot(18) - 2\,(6) = -15-36-12 = -63$

Thm: If T is triangular, ~~they~~ $T = \begin{bmatrix} t_{11} & * & \cdots & * \\ 0 & t_{22} & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$ or $T = \begin{bmatrix} t_{11} & 0 & \cdots & 0 \\ * & t_{22} & & \vdots \\ * & * & \ddots & 0 \\ * & \cdots & * & t_{nn} \end{bmatrix}$

(upper Δ) (lower Δ)

then $\det T = t_{11} t_{22} \cdots t_{nn}$ = prod of diagonal entries.

Example $\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 2 & 10 & 4 \end{bmatrix} = 1 \cdot (-2) \cdot 4 = -8$ & $\det \begin{bmatrix} 0 & -3 & 7 \\ 0 & 2 & 8 \\ 0 & 0 & 10 \end{bmatrix} = 0 \cdot 2 \cdot 10 = 0$

- We care about determinants because they tell us when a matrix is singular (noninvertible)

Thm: A of size $n \times n$ is singular if and only if $\det A = 0$.

We saw this early for 2×2 matrices. The result is more general!