Lecture XXVIII §6.3 Elementary Operations & Determinants

Recall: Given a matrix A of size $n\times n$, we define the determinant of A as

$$\det A = a_{11}(-1)^{1+1}\det M_{1,1} + a_{12}(-1)^{1+2}\det M_{1,2} + \cdots + a_{1n}(-1)^{1+n}\det M_{1,n}$$

where $M_{r,s}$ = matrix of size $(n-1)\times(n-1)$ obtained by removing row r & col s from A.

Eg
$$\det\begin{pmatrix}1&0&2\\1&1&2\\3&4&5\end{pmatrix} = 1\cdot(-1)^{1+1}\det\begin{bmatrix}1&2\\4&5\end{bmatrix} + 0 + 2\cdot(-1)^{1+3}\det\begin{bmatrix}1&1\\3&4\end{bmatrix}$$
$$= 1\,(5-8) + 2\,(4-3) = -3+2 = \boxed{-1}$$

- **Thm**: A is singular if and only if $\det A = 0$.
- When A is triangular $\left(\begin{pmatrix}a_{11}& & *\\ &\ddots& \\ 0 & & a_{nn}\end{pmatrix} \text{ or } \begin{pmatrix}a_{11}& & 0\\ &\ddots& \\ * & & a_{nn}\end{pmatrix}\right)$ then $\det A = a_{11}\cdots a_{nn}$

TODAY: Use elementary row operations to simplify the calculation of determinants

3 row operations $\longleftrightarrow$ 3 effects on determinants

§1 Operation 1: Exchange 2 rows.

Thm 1: Write B for the matrix obtained from A by exchanging Rows i & j. Then $\det(B) = -\det A$.

Example
$$A = \begin{pmatrix}1&0&2\\1&1&2\\3&4&5\end{pmatrix} \xrightarrow[R_1\leftrightarrow R_2]{} B = \begin{pmatrix}1&1&2\\1&0&2\\3&4&5\end{pmatrix}$$

$$\det B = 1\,(-1)^{1+1}\det\begin{pmatrix}0&2\\4&5\end{pmatrix} + 1\,(-1)^{1+2}\det\begin{pmatrix}1&2\\3&5\end{pmatrix} + 2\,(-1)^{1+3}\det\begin{pmatrix}1&0\\3&4\end{pmatrix}$$
$$= 1\,(0-8) - 1\,(5-6) + 2\,(4-0) = -8+1+8 = 1 = \underline{-\det A}\ \checkmark$$

§2 Operation 2: Multiplying a row by a nonzero scalar α

Thm 2: Write B for the matrix obtained from A by multiplying Row i by scalar $\alpha \neq 0$. Then $\det(B) = \alpha\det(A)$.

Example:
$$A = \begin{pmatrix}1&0&2\\1&1&2\\3&4&5\end{pmatrix} \xrightarrow[R_2\to 3R_2]{} B = \begin{pmatrix}1&0&2\\3&3&6\\3&4&5\end{pmatrix}$$

$$\det B = 1\,(-1)^{1+1}\det\begin{vmatrix}3&6\\4&5\end{vmatrix} + 0 + 2\,(-1)^{1+3}\det\begin{vmatrix}3&3\\3&4\end{vmatrix}$$

$= 1 \cdot (15-24) + 2\,(12-9) = -9+6 = -3 = 3(-1) = 3 \det A$

Note: $\det \begin{vmatrix} 3 & 6 \\ 4 & 5 \end{vmatrix} = 3 \det \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$ & $\det \begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = 3 \det \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix}$

In general, if scalar multiplies row 1, we see the factor in each b_{1i}

• ——————— a different row ——————— the smaller determinants.

Consequence: If we scale the matrix by a scalar c, then

$$\det(cA) = c^n \det A \quad \text{if } A \text{ has size } n \times n$$

[Why? Because we scaled each row by c, so we get one factor per row of A]

§3 Operation 3: Replace Row i by $\text{Row}_i + c\,\text{Row}_j$ with $j \neq i$, c any scalar.

Thm 3: Write B for the matrix obtained from A by replacing Row_i by $\text{Row}_i + c\,\text{Row}_j$ for a scalar c, where $i \neq j$. Then $\det(B) = \det(A)$

Example $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[R_2 \to R_2 - R_1]{} B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 4 & 5 \end{pmatrix}$

$$\det(B) = 1\,(-1)^{1+1} \det \begin{pmatrix} 1 & 0 \\ 4 & 5 \end{pmatrix} + 0 + 2\,(-1)^{1+3} \det \begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix}$$
$$= (5-0) + 2(-3) = 5-6 = -1 = \det A \quad \checkmark$$

§4 Combine all operations

Algorithm (To compute $\det A$): ① Use row operations (keeping track on their effects on determinants) until we get a triangular matrix T in the end.

② Computing $\det(T)$ is easy!

③ Use the recorded operations to compute $\det(A)$ (backtracking!)

Example: $A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 4 & 3 & 3 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix}$ Compute $\det A$.

• Use row-reduction to ~~push~~ A into a triangular matrix

• At each step $A \to B$, write $\det(B)$ in terms of $\det A$

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 4 & 3 & 3 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 2R_1]{R_3 \to R_3 + 3R_1} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 4 & 3 & 3 \\ 0 & 8 & -1 & 6 \\ 0 & -7 & -2 & -3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 4 & 3 & 3 \\ 0 & 0 & -7 & 0 \\ 0 & -7 & -2 & -3 \end{bmatrix}$$

det A [Operation 3 keeps the det] det (A) — no changes! — det(A)

$$\xrightarrow{R_2 \to \frac{R_2}{4}} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 3/4 & 3/4 \\ 0 & 0 & -7 & 0 \\ 0 & -7 & -2 & -3 \end{bmatrix} \xrightarrow{R_4 \to R_4 + 7R_2} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 3/4 & 3/4 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & \frac{13}{4} & \frac{9}{4} \end{bmatrix}$$

$$-2 + \frac{7\cdot 3}{4} = \frac{21-8}{4} = \frac{13}{4}$$

$$-3 + \frac{7\cdot 3}{4} = \frac{21-12}{4} = \frac{9}{4}$$

factor $= \frac{1}{4}$ — $\frac{1}{4}(\det A)$ — no changes! — $\frac{1}{4} \det A$

$$\xrightarrow{R_4 \to 4R_4} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 3/4 & 3/4 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 13 & 9 \end{bmatrix} \xrightarrow{R_3 \to \frac{R_3}{7}} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 3/4 & 3/4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 13 & 9 \end{bmatrix} \xrightarrow{R_4 \to R_4 + 13R_3} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 3/4 & 3/4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix} = T$$

factor 4 — $4 \cdot \frac{1}{4} \det A = \det A$ — factor $\frac{1}{7}$ — $\frac{1}{7} \det A$ — no changes — $\frac{1}{7} \det A$

We have $\det T = \frac{1}{7} \det A$

And $\det T$ = prod of diagonal entries $= 1 \cdot 1 \cdot (-1) \cdot 9 = -9$

<u>Conclusion:</u> $\det A = 7 \det T = 7(-9) = -63.$

<u>Aside</u>: Matrix from lecture XXVII: $A' = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & 2 & 3 & 1 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_1 + R_2]{} A$

So we know $\det A' = -63$ as we computed last time!