Lecture XXIX : §6.4 Cramer's Rule

Recall: Row operations change determinants in specific ways

(I) $A \xrightarrow[R_i \leftrightarrow R_j]{} B$ $\quad \det(B) = -\det(A)$

(II) $A \xrightarrow[R_i \to cR_i]{} B$ $\quad c \neq 0 \quad \det(B) = c\det(A)$

(III) $A \xrightarrow[R_i \to R_i + cR_j]{} B$ $\quad i \neq j \quad \det(B) = \det(A)$

$\leadsto$ Algorithm to compute $\det(A)$ (make it triangular under these 3 operations)

Thm 1: $\det(A) = \det(A^T)$

$\Rightarrow$ We can perform operations between columns of A & get similar rules for the effect of elementary column operations on determinants

§1 Product rule:

Thm 2: $\boxed{\det(AB) = \det(A)\det(B)}$ (This will shows why A singular if and only if $\det A = 0$)

- So if A is invertible, then $\underset{(\det A)(\det A^{-1})}{\det(AA^{-1})} = \det I_n = 1$ so $\det A^{-1} = \frac{1}{\det A}$ & $\det A \neq 0$

Example: $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$ $\quad$ ~~and~~ $\quad AB = \begin{bmatrix} 0 & 2 \\ 2 & -3 \end{bmatrix}$

$\det A = 6 - 2 = 4$ $\quad \det B = 2 - 3 = -1$ $\quad \det(AB) = -4$.

- If A is singular, then $A \sim B = \begin{bmatrix} * & & * \\ 0 & \ddots & \\ 0 & \cdots & 0 \end{bmatrix}$ where the matrix is in REF & at least the last row $= 0$.

so $\det B = k \det(A)$ for some $k \neq 0$ (using rules (I)–(III))

But $\det(B) = 0$ (triangular with $0 = b_{nn}$) & $k \neq 0$, so $\det A = 0$.

Q: How to prove Product Rule?

Lemma: If A is singular, $n \times n$ matrix & B is any $n \times n$ matrix, then AB is also singular & so $\det(AB) = 0 = \underbrace{\det A}_{=0} \det B$

Proof Recall that AB is singular if and only if $(AB)^T = B^T A^T$ is singular.

Now, if A is singular, so is A^T & $N(A^T)$ contains a nonzero vector v

Then $B^T A^T v = B^T \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ so $N(B^T A^T)$ contains $v \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ &

so $B^T A^T$ is singular. We conclude AB is singular. $\square$

Proof of the Product Rule:

- If A is singular, the result follows by the Lemma.
- We assume A is nonsingular, so $A \underset{\text{row-equiv}}{\sim} I_n$ (Recall $(A|I) \sim (I|A^{-1})$)

Then $1 = \det I_n = k \det A$. We get $\det A = \frac{1}{k}$.

But the same row operations give $AB \sim I_n B$ so $\det(I_n B) = k \det(AB)$

Then
$$\det B = k \det AB$$
$$\frac{1}{k} \det B = \det AB$$
$$\boxed{\det A \det B = \det AB}$$

Example: $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

$$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \underset{-\det A}{\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}} \xrightarrow{R_2 \to R_2 - 2R_1} \underset{-\det A}{\begin{pmatrix} 1 & 3 \\ 0 & -4 \end{pmatrix}} \xrightarrow{R_2 \to \frac{R_2}{-4}} \underset{\underbrace{\left(\frac{-1}{4}\right)(-\det A)}_{\frac{\det A}{4}}}{\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}}$$

$$\xrightarrow{R_1 \to R_1 - 3R_2} \underset{\frac{\det A}{4}}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

Use the same row operations to transform AB into B

$$AB = \begin{bmatrix} 0 & 2 \\ 2 & -3 \end{bmatrix} \xrightarrow{R_1 \to R_2} \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 2 & -3 \\ -4 & 8 \end{pmatrix} \xrightarrow{R_2 \to \frac{R_2}{-4}} \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \xrightarrow{R_1 \to R_1 - 3R_2} \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix} = B \checkmark$$

§2 Cramer's Rule:

GOAL: Use determinants to find the unique solution to $A\underline{x} = \underline{b}$ when A is a nonsingular $n \times n$ matrix.

Thm 3 [Cramer's Rule] Fix $A = [A_1 \cdots A_n]$ a nonsingular $n \times n$ matrix, and a vector $\underline{b}$ in $\mathbb{R}^n$. For each $i = 1, \ldots, n$, we build a new matrix $B_i = [A_1 \cdots A_{i-1}\ b\ A_{i+1} \cdots A_n]$ by replacing the i^{th} column of A with $\underline{b}$.

Then, the unique solution $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ to $A\underline{x} = b$ satisfies:
$$x_1 = \frac{\det(B_1)}{\det(A)}, \quad x_2 = \frac{\det(B_2)}{\det A}, \quad \ldots, \quad x_n = \frac{\det(B_n)}{\det A}.$$

Proof for x_1:

We know $x_1 A_1 + x_2 A_2 + \cdots + x_n A_n = A \cdot \underline{x} = \underline{b}$

We transpose: $x_1 A_1^T + \cdots + x_n A^T = \underline{x}^T A^T = \underline{b}^T$

We write a new matrix C by multiplying Row_1 of A^T by x_1

$$C = \begin{bmatrix} x_1 A_1^T \\ A_2^T \\ \vdots \\ A_n^T \end{bmatrix} \quad \text{so} \quad \boxed{\det C = x_1 \det A^T \underset{\text{by Thm 1}}{=} x_1 \det A}$$

But $x_1 A_1^T = \underline{b}^T - x_2 A_2^T - \cdots - x_n A_n^T$

So
$$\det C = \det\begin{bmatrix} b^T - x_2 A_2^T - \cdots - x_n A_n^T \\ A_2^T \\ \vdots \\ A_n^T \end{bmatrix} \underset{\substack{\text{by def of det} \\ \text{linear on any row}}}{=} \det\begin{bmatrix} b^T \\ A_2^T \\ \vdots \\ A_n^T \end{bmatrix} - x_2 \det\begin{bmatrix} A_2^T \\ A_2^T \\ \vdots \\ A_n^T \end{bmatrix} - x_3 \det\begin{bmatrix} A_3^T \\ A_2^T \\ A_3^T \\ \vdots \\ A_n^T \end{bmatrix} - \cdots - x_n \det\begin{bmatrix} A_n^T \\ \vdots \\ A_n^T \end{bmatrix}$$

But all the matrices $\begin{bmatrix} A_i^T \\ A_2^T \\ \vdots \\ A_n^T \end{bmatrix}$ $i=2,\dots,n$ have 2 repeated rows, so they are singular!

so
$$\boxed{\det C} = \det\begin{bmatrix} b^T \\ A_2^T \\ \vdots \\ A_n^T \end{bmatrix} = \det B_1^T \underset{\text{by Thm 1}}{=} \boxed{\det B_1}.$$

We get $x_1 \det A = \det C = \det B_1$

<u>Example</u> Solve $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{b}$ $\quad A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ for $\underline{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ using Cramer's Rule.

$\det A = 4$

(1) $B_1 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \rightsquigarrow \det B_1 = 3-4 = -1 \rightsquigarrow x_1 = \dfrac{\det B_1}{\det A} = \dfrac{-1}{4}$

$B_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightsquigarrow \det B_2 = 4-1 = 3 \rightsquigarrow x_2 = \dfrac{\det B_2}{\det A} = \dfrac{3}{4}$

Check: $A\underline{x} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1/4 \\ 3/4 \end{bmatrix} = \begin{bmatrix} \frac{-2+6}{4} \\ \frac{-1+9}{4} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \underline{b}$ ✓

(2) $B_1 = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \rightsquigarrow \det B_1 = 9-8 = 1 \rightsquigarrow x_1 = \dfrac{1}{4}$

$B_2 = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \rightsquigarrow \det B_2 = 8-3 = 5 \rightsquigarrow x_2 = \dfrac{5}{4}$

<u>Check</u>: $Ax = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1/4 \\ 5/4 \end{bmatrix} = \begin{bmatrix} \frac{2+10}{4} \\ \frac{1+15}{4} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \underline{b}$ ✓