Lecture XXX : §4.1-4.2 The Eigenvalue Problem for (2×2) matrices

§1 The eigenvalue problem: (German: eigen = "self")

(EV) $\begin{cases} \text{Fix an } n \times n \text{ matrix } A. \text{ Find } \underline{\text{all}} \text{ scalars } \lambda \text{ in } \mathbb{R} \text{ such that} \\ \qquad A\underline{x} = \lambda \underline{x} \\ \text{has a nonzero solution } \underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}. \end{cases}$

Definition: Scalars satisfying (EV) are called __eigenvalues of A__. The solutions $\underline{x}$ to $A\underline{x} = \lambda \underline{x}$ are called __eigenvectors__ of the eigenvalue λ.

__Why is (*) relevant?__ Use eigenvalues to:

(1) solve differential equations,
(2) analyze population growth,
(3) calculate powers of matrices,
(4) diagonalize linear transformations $T: \mathbb{R}^n \to \mathbb{R}^n$.
(5) simplify and draw quadratic forms: $Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$
A, B, C, D, E, F fixed real numbers.

Diagonalizing $T: \mathbb{R}^n \to \mathbb{R}^n$:

- Fix those vectors v in $\mathbb{R}^n$, where $T(v) = \lambda v$ for some λ.
- Eg $\lambda = 0$, then these vectors form the NullSpace of T

If we can find a basis $B = \{v_1, \dots, v_n\}$ of $\mathbb{R}^n$ where $T(v_i) = \lambda_i v_i$ for some λ_i

then $[T]_{BB} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \lambda_n \end{bmatrix}$ so it's a diagonal matrix (simplest kind of transformations!)

Since $v_i \neq 0$, & $T(v_i) = \lambda_i v_i$, then λ_i is an eigenvalue for T.

- So to "diagonalize T" we need to find the eigenvalues.

Remark: $A\underline{x} = \lambda \underline{x}$ always has $\underline{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ as a solution, so to find eigenvalues we need to search for $\underline{x} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ solving (EV)

§2. Strategy to solve the (EV) problem for λ: A is a fixed $n\times n$ matrix.

Notice: $A\underline{x} = \lambda \underline{x}$ has a nontrivial solution $\underline{x}$ is equivalent to
$A\underline{x} - \lambda\underline{x} = \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$ ________________.

But $\underbrace{A}_{\text{matrix}}\underline{x} - \underbrace{\lambda}_{\text{scalar}}\underline{x} = A\underline{x} - \lambda I_n \underline{x} = (A-\lambda I_n)\underline{x}$

so (EV) is equivalent to $(A-\lambda I_n)\underline{x} = \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$ has a nonzero solution $\underline{x}$

This is the SAME as saying $(A-\lambda I_n)$ is a singular $n\times n$ matrix.

STRATEGY:

(1) Find all λ's where $A-\lambda I_n$ is singular (at most n of them)

[How? $\det(A-\lambda I_n) = 0$]

(2) Given all the finitely many λ's from (1), find all $\underline{x}$ solving
$(A-\lambda I_n)\underline{x} = \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$ by Gauss-Jordan elimination!

$\rightsquigarrow$ NullSpace $(A-\lambda I_n) = \{\underline{x} : (A-\lambda I_n)\underline{x} = \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}\} =: E_\lambda$

is the Eigenspace of the eigenvalue λ.

§3. The EV problem for 2x2 matrices:

Write $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$. Then:

$$A-\lambda I_2 = \begin{bmatrix}a & b\\ c & d\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} = \begin{bmatrix}a-\lambda & b\\ c & d-\lambda\end{bmatrix}$$

- $A-\lambda I_2$ is singular if and only if $\det(A-\lambda I_2) = 0$
- We write the determinant explicitly:

$$\det\begin{pmatrix}a-\underline{\underline{\lambda}} & b\\ c & d-\underline{\underline{\lambda}}\end{pmatrix} = (a-\underline{\underline{\lambda}})(d-\underline{\underline{\lambda}}) - bc = ad - a\underline{\underline{\lambda}} - \underline{\underline{\lambda}}d + \underline{\underline{\lambda}}^2 - bc$$
$$= \lambda^2 + (-a-d)\lambda + \underbrace{ad-bc}_{=\det A} = 0$$

- How do we solve $\lambda^2 + p\lambda + q = 0$ for λ? Use quadratic formula $\lambda = \dfrac{-p \pm\sqrt{p^2-4q}}{2}$

Example 1: $A = \begin{bmatrix} 5 & -1 \\ 8 & -1 \end{bmatrix}$ Find the eigenvalues of A & the corresponding eigenspaces.

$\det(A - \lambda I_2) = \lambda^2 + (-5+1)\lambda + (-5+8) = \lambda^2 - 4\lambda + 3 = 0$

Solns: $\lambda = \frac{4 \pm \sqrt{16 - 4 \cdot 3}}{2} = \frac{4 \pm \sqrt{4}}{2} = \frac{4 \pm 2}{2} = 2 \pm 1$ ↗ 3, ↘ 1.

Or Directly = factor the polynomial $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$.

- Eigenspaces? For each λ: $E_\lambda = \{ \underline{x} : (A - \lambda I_2)\underline{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}$.

$\underline{\underline{E_3}}$: $(A - 3I_2)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$A - 3I_2 = \begin{bmatrix} 5-3 & -1 \\ 8 & -1-3 \end{bmatrix}$ Use GJ to find the solutions!

$\begin{bmatrix} 2 & -1 \\ 8 & -4 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 4R_1]{} \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad 2x_1 - x_2 = 0$ $\quad \boxed{2x_1 = x_2}$ $\quad$ Soln: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

So $E_3 = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$ $\quad$ Check $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5-2 \\ 8-2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ✓

$\underline{\underline{E_1}}$: $(A - I_2) = \begin{bmatrix} 5-1 & -1 \\ 8 & -1-1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 8 & -2 \end{bmatrix}$ $\quad$ Solve $\begin{bmatrix} 4 & -1 \\ 8 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 4 & -1 \\ 8 & -2 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{} \begin{bmatrix} 4 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad 4x_1 - x_2 = 0$ $\quad \boxed{4x_1 = x_2}$ $\quad$ Soln $= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

so $E_1 = \text{Span}\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix} \right)$ $\quad$ Check: $A\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5-4 \\ 8-4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ ✓

$\rightsquigarrow$ Basis for $\mathbb{R}^2$ of eigenvectors $= \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$

Example 2: $A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ Find the eigenvalues & eigenspaces.

$\det(A - \lambda I_2) = \begin{vmatrix} 3-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) = (\lambda - 3)(\lambda + 1) = 0$

Solns: $\lambda = 3$, $\lambda = -1$.

$\underline{\underline{E_3}}$: $A - 3I_2 = \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}$ $\rightsquigarrow$ soln: x_1 any, $x_2 = 0$ $\quad E_3 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$\underline{\underline{E_{-1}}}$: $A - (-I_2) = \begin{bmatrix} 3+1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$ $\rightsquigarrow$ soln $x_1 = 0$, x_2 any $\quad E_{-1} = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

Check: $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.