Lecture XXXI : §4.4 Eigenvalues and the Characteristic Polynomial

Recall : The (EV) Problem : Given an $n \times n$ matrix A, we want to find ALL values of $\boxed{\lambda}$ ($\to$ SCALAR) (called Eigenvalues) for which : $A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \boxed{\lambda}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ has a solution $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ (we call $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ an eigenvector with eigenvalue λ)

Last Time : we saw this is equivalent to finding all values of λ when

$$A - \lambda I_n = A - \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda \end{bmatrix}$$ is a singular $n \times n$ matrix

Q: How can we check this? Use determinants!

$A - \lambda I_n$ is singular if and only if $\det(A - \boxed{\lambda} I_n) = 0$

$$\det(\underset{\substack{\uparrow \\ \text{fixed}}}{A} - \boxed{\lambda} I_n) = 0$$

Example 0: $A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$

$$A - \lambda I_3 = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & -2 \\ 1 & 3-\lambda & 1 \\ 1 & 3 & 1-\lambda \end{bmatrix} \leftarrow$$

Compute its determinant using the definition:

$$\det\begin{pmatrix} 1-\lambda & 0 & -2 \\ 1 & 3-\lambda & 1 \\ 1 & 3 & 1-\lambda \end{pmatrix} = (1-\lambda)(-1)^{1+1}\det\begin{pmatrix} 3-\lambda & 1 \\ 3 & 1-\lambda \end{pmatrix} + 0 + (-2)(-1)^{1+3}\det\begin{pmatrix} 1 & 3-\lambda \\ 1 & 3 \end{pmatrix}$$

$$= (1-\lambda)\Big((3-\lambda)(1-\lambda) - 3\Big) - 2\Big(3 - (3-\lambda)\Big)$$

$$= (1-\lambda)(3 - 3\lambda - \lambda + \lambda^2 - 3) - 2\lambda$$

$$= (1-\lambda)(\lambda^2 - 4\lambda) - 2\lambda$$

$$= (1-\lambda)\lambda(\lambda-4) - 2\lambda$$

$$= \lambda\big((1-\lambda)(\lambda-4) - 2\big) = \lambda(-\lambda^2 + 5\lambda - 4 - 2)$$

$$= \lambda(-\lambda^2 + 5\lambda - 6) = -\lambda(\lambda^2 - 5\lambda + 6) \nearrow \text{degree 3 polynomial in } \lambda$$

- Roots of $\lambda^2 - 5\lambda + 6$ are 2 & 3 since $\lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$

(**Alternatively** : solve with the quadratic formula!)

So the eigenvalues are : $\lambda = 0$, $\lambda = 2$ & $\lambda = 3$.

- Next, we give a name to $\det(A - \lambda I_n)$ & study its properties:

§1. The Characteristic Polynomial:

Definition: $P_A(t) = \det(A - \textcircled{t}\, I_n)$ is called the characteristic polynomial of A (in the variable $\textcircled{t}$)

From the examples of 2×2 & 3×3 matrices we can derived a general result about $P_A(t)$.

Thm 1: $P_A(t)$ is a polynomial of degree n in t (n = size of the matrix A)

Why? t's appear in the diagonal of $\tilde{A} = A - \begin{pmatrix} t & & 0 \\ & \ddots & \\ 0 & & t \end{pmatrix}$ & using the definition of det, we see that each summand in the expression of $\det(\tilde{A})$ is a polynomial in λ of degree at most n, & the summand for the index (1,1) has degree exactly n.

Thm 2: The eigenvalues of A are the roots of the polynomial $P_A(t)$

Natural questions:
- How many roots does $P_A(t)$ have? At most n
- How can we find them? Some heuristic methods...

- Example 1: $A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$

$$P_A(t) = \det \begin{bmatrix} -2-t & -1 \\ 1 & -2-t \end{bmatrix} = \underbrace{(-2-t)^2}_{\geq 0 \text{ if } t \text{ is a real number}} + 1 = t^2+4t+5$$

, so this polynomial has no real roots!

- Example 2: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 1 & -2 \end{bmatrix}$

$$P_A(t) = \det \begin{bmatrix} 1-t & 0 & 0 \\ 0 & -2-t & -1 \\ 0 & 1 & -2-t \end{bmatrix} = (1-t)(t^2+4t+5)$$

so 1 real root

- Example 3: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

$$P_A(t) = \det \begin{bmatrix} 1-t & 0 & 0 \\ 0 & 2-t & 0 \\ 0 & & -3-t \end{bmatrix} = (1-t)(2-t)(-3-t) = -(t-1)(t-2)(t+3)$$

so 3 roots!

Note: If we allow roots to be complex numbers, then $P_A(t)$ has exactly n roots in the complex numbers (counted with multiplicity).

Example 1 (revisited) $\det(A - tI_2) = t^2+4t+5$

Roots: $\dfrac{-4 \pm \sqrt{4^2-4\cdot 5}}{2} = \dfrac{-4 \pm \sqrt{16-20}}{2} = \dfrac{-4 \pm \sqrt{-4}}{2}$

Complex numbers provide roots of negative numbers, by adding a root of -1, which we call i.

So $\sqrt{-4} = \sqrt{4(-1)} = 2i$

Then, roots of $P_A(t)$ are $\dfrac{-4 \pm 2i}{2} = -2 \pm i$ $\begin{cases} -2+i \\ -2-i. \end{cases}$

- We'll see more about this in Section §4.6 (Lecture XXXIV)

§2 Properties of Eigenvalues:

Thm 3: Fix A an $n\times n$ matrix & an eigenvalue λ on it. Then:

(1) λ^k is an eigenvalue of A^k for $k=2,3,4,\ldots$

(2) If A is nonsingular, then $\frac{1}{\lambda}$ is an eigenvalue for A^{-1}

(3) If μ is any scalar, then $\lambda+\mu$ is an eigenvalue of $\tilde{A} = A+\mu I_n$

And they all share the same eigenvector (eigenvector for A)

Proof: (1) Do it for $k=2$.

By definition $A\underline{x} = \lambda \underline{x}$ for some $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

Then: multiply by A: $A(A\underline{x}) = A\lambda\underline{x} = \lambda A x = \lambda(\lambda\underline{x}) = \lambda^2 x$

So $A^2 X = \boxed{\lambda^2} \underline{X}$. has a solution $\underline{x} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

We see that λ^2 is an eigenvalue for A^2 (& same eigenvector!)

(2) A is nonsingular, then $A\underline{x} = \underline{0}$ has no nontrivial solution, so $\lambda \neq 0$ & we can invert it!

Write $A\underline{x} = \lambda\underline{x}$ for some $\underline{x} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ & multiply by A^{-1}!

Then $A^{-1}(A\underline{x}) = A^{-1}\lambda\underline{x}$

$\underline{x} = \lambda A^{-1}\underline{x}$ so $A^{-1}\underline{x} = \boxed{\frac{1}{\lambda}}\underline{x}$ for $\underline{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

so $\frac{1}{\lambda}$ is an eigenvalue for A^{-1}. (& same eigenvector!)

SAME eigenvector!

(3) Write $A\underline{x} = \lambda\underline{x}$ for some $\underline{x} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $(A+\mu I_n)\underline{x} = A\underline{x} + \mu \underline{x} = (\lambda+\mu)\underline{X}$

& $x \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Example 0 (revisited) $A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ $P_A(t) = -t(t-2)(t-3)$

- A is singular because $\lambda = 0$ is an eigenvalue. (Equivalent conditions!)
- Eigenvalues for A^2 include: $0^2, 2^2, 3^2$ $\rightsquigarrow$ so they are <u>all</u>!
- Eigenvalues for $A - I_3$ include: $0-1, 2-1, 3-1$ $\underset{\uparrow}{\rightsquigarrow}$ ________ !

[at most 3 eigenvalues]