Lecture XXXII: §4.5 Eigenvectors & eigenspaces

Recall: Given A of size $n\times n$, the characteristic polynomial of A is

$P_A(t) = \det(A - tI_n)$ & satisfies: • degree in $t = n$
• The eigenvalues of A are the roots of $P_A(t)$.

Properties: If λ is an eigenvalue of A, then

(1) If A is invertible, then $\lambda \neq 0$ & $1/\lambda$ is an eigenvalue of A^{-1}

(2) λ^k is an eigenvalue of A^k for $k = 2, 3, \ldots$

(3) for any μ scalar: $\lambda+\mu$ is an eigenvalue of $A+\mu I_n$

[The same eigenvector $\underline{x}$ of A works for the other matrices]

Thm: A & A^T share the same eigenvectors.

Proof: $P_{A^T}(t) = P_A(t)$ because $(A^T - \underline{t}I_n) = (A - \underline{\underline{t}}I_n)^T$ symmetric. scalar

& $\det((A-tI_n)^T) = \det(A - tI_n)$.

§1 Eigenvectors:

TODAY: Focus on eigenvectors! (TODAY: Only focus on real eigenvalues)

Given λ eigenvalue of A, use Gauss-Jordan elimination to

$$\boxed{\text{Solve } \tilde{A}\underline{x} = \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix} \quad \text{for } \tilde{A} = A - \lambda I_n.}$$

Example 1: $A = \begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}$

$P_A(t) = \det\begin{pmatrix}-1-t & 1\\ 0 & -1-t\end{pmatrix} = (-1-t)^2 = (t+1)^2$ so $\lambda = -1$ is the only eigenvalue of A (multiplicity $= 2$)

$E_{-1} = \{ \underline{x} = \begin{bmatrix}x_1\\ x_2\end{bmatrix} : A\underline{x} = (-1)\underline{x}\}$ is the collection of eigenvectors with eigenvalue -1.

$\tilde{A} = A - (-1)I_2 = \begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix} - \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ solutions: $x_2 = 0$, x_1 any

So $E_{-1} = \{ \underline{x} = \begin{bmatrix}x_1\\ 0\end{bmatrix} = x_1\begin{bmatrix}1\\ 0\end{bmatrix} : x_1 \text{ any scalar}\} = \mathrm{Span}\left(\begin{bmatrix}1\\ 0\end{bmatrix}\right)$

• One eigenvalue (-1) of algebraic multiplicity 2 as a root of $P_A(t)$ but only 1 generator for E_{-1}, so we don't have a basis for $\mathbb{R}^2$ consisting of ONLY eigenvectors.

Example 2: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$$P_A(t) = \det\begin{pmatrix} 2-t & 0 & 0 \\ 0 & -1-t & 1 \\ 0 & 0 & -1-t \end{pmatrix} = (2-t)(-1-t)^2 \longrightarrow \lambda = 2, \ \lambda = -1$$

eigenvalues (multiplicity 2)

- E_{-1}: $\tilde{A} = A - (-1)I_3 = A + I_3 = \begin{bmatrix} 2+1 & 0 & 0 \\ 0 & -1+1 & 1 \\ 0 & 0 & -1+1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ rk 2

so $\begin{cases} 3x_1 = 0 \\ x_2 \text{ any} \\ x_3 = 0 \end{cases}$ $\rightsquigarrow$ one generator for E_{-1}: $\begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad E_{-1} = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

- E_2: $\tilde{A} = A - 2I_3 = \begin{bmatrix} 2-2 & 0 & 0 \\ 0 & -1-2 & 1 \\ 0 & 0 & -1-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

Use GJ: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & -3 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & -3 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

So $\begin{cases} -3x_2 = 0 \\ x_3 = 0 \\ x_1 \text{ any} \end{cases}$ $\rightsquigarrow$ $\underline{x} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad E_2 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

- Two eigenvalues: 2 & (-1), with multiplicities 1 & 2 respectively as roots of $P_A(t)$. Only 1 generator per eigenvalue, so again we don't have a basis for $\mathbb{R}^3$ consisting of eigenvectors.

Example 3: $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 8 \\ 0 & 0 & 4 \end{bmatrix}$ $\rightsquigarrow$ $P_A(t) = \det\begin{pmatrix} 2-t & 0 & 1 \\ 0 & -1-t & 8 \\ 0 & 0 & 4-t \end{pmatrix} = (4-t)(-1-t)(2-t)$

3 eigenvalues: $\lambda = 4, -1$ & 2.

- E_{-1}: $\tilde{A} + I_3 = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 0 & 8 \\ 0 & 0 & 5 \end{bmatrix}$ rank=2 solutions $x_2 = 0$ $\quad E_{-1} = \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$
- E_2: $\tilde{A} - 2I_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -3 & 8 \\ 0 & 0 & -2 \end{bmatrix}$ rank=2 Solutions $x_1 = 0$ $\quad E_2 = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$
- E_4: $\tilde{A} - 4I_3 = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -5 & 8 \\ 0 & 0 & 0 \end{bmatrix}$ rank=2 $\begin{matrix} -2x_1 + x_3 = 0 \\ -5x_2 + x_3 = 0 \end{matrix}$ $\rightsquigarrow$ $x = \begin{bmatrix} +x_3/2 \\ +x_3/5 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1/2 \\ 1/5 \\ 1 \end{bmatrix}$

So $E_4 = \text{Span}\left(\begin{bmatrix} 1/2 \\ 1/5 \\ 1 \end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix} 5 \\ 2 \\ 10 \end{bmatrix}\right)$

Claim: $B = \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 10 \end{bmatrix}\right\}$ is a basis for $\mathbb{R}^3$, & consists of eigenvectors.

(clearly l.i & have $3 = \dim \mathbb{R}^3$)

§2 Eigenspaces:

Given λ an eigenvalue of A, we write:

$$E_\lambda = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : (A - \lambda I_n)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

Prop: $E_\lambda = \text{NullSpace}(A - \lambda I_n)$ so it is a subspace of $\mathbb{R}^n$.

We call it the eigenspace of A with corresponding eigenvalue λ.

Q: What is $\dim E_\lambda$? We define it as the geometric multiplicity of λ

Contrast: Algebraic multiplicity of λ = multiplicity of λ as a root of $P_A(t)$

Earlier examples:

Example 1 $\lambda = -1$ has algebraic multiplicity 2 & geometric mult = 1

because $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ has $P_A(t) = (t+1)^{2}$ & $E_{-1} = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$. dim = 1

Example 2: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ $P_A(t) = (2-t)(t+1)^2$, $E_{-1} = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$, $E_2 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

- $\lambda = 2$ has alg. mult = 1 & geom. mult = 1.
- $\lambda = -1$ __________ = 2 & __________.

Example 3: $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 8 \\ 0 & 0 & 4 \end{bmatrix}$ $P_A(t) = (4-t)(-1-t)(2-t)$

$E_{-1} = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$, $E_2 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$. $E_4 = \text{Span}\left(\begin{bmatrix} 5 \\ 2 \\ 10 \end{bmatrix}\right)$

- $\lambda = -1$ has alg mult = 1 & geom. mult = 1.
- $\lambda = 2$ ____________________.
- $\lambda = 4$ ____________________

Notice (from examples):
- alg mult $(\lambda) \geq 1$, geom. mult$(\lambda) \geq 1$ (NullSpace$(A - \lambda I_n) \neq \{0\}$)
- alg mult $(\lambda) \geq$ geom. mult (λ) (HARD Theorem!)

We'll discuss these statements next time.