Lecture XXXIII : §4.5 (cont) Eigenvectors & eigenspaces

Recall: Given λ eigenvalue of an $n\times n$ matrix A, we have:

$$E_\lambda = \text{NullSpace}(A-\lambda I_n) = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right\}$$

is a subspace of $\mathbb{R}^n$ and $E_\lambda \neq \{0_V\}$ whenever λ is an eigenvalue (by definition!)

Define: • algebraic multiplicity of λ = order of λ as a root of $P_A(t)$

• geometric multiplicity of λ = dimension $(E_\lambda) \geq 1$.

We are interested in the case where alg mult (λ) = geom mult (λ) for all eigenvalues λ.

In that case, we'll be able to diagonalize A.

We first discuss the bad cases, called defective.

§1 Defective matrices:

Def: Fix an $n\times n$ matrix A. If A has an eigenvalue λ (real or complex root of $P_A(t)$) where alg mult (λ) > geom mult (λ), we call A a <u>defective matrix</u>.

• Last time: out of the 3 examples, first 2 where defective.

• More examples:

Example: $A = \begin{bmatrix} -3 & 1 & 1 & 1 \\ -16 & 4 & 4 & 4 \\ -7 & 2 & 3 & 1 \\ -11 & 1 & 3 & 5 \end{bmatrix}$ Find the eigenvalues of A & determine the algebraic and geometric multiplicities of each of them.

Compute the characteristic polynomial by row/column operations

$$P_A = \det(A - tI_4) = \det\begin{bmatrix} -3-t & 1 & 1 & 1 \\ -16 & 4-t & 4 & 4 \\ -7 & 2 & 3-t & 1 \\ -11 & 1 & 3 & 5-t \end{bmatrix} \underset{\substack{col_2 \to col_2 - col_4 \\ col_3 \to col_3 - col_4}}{=} \det \begin{bmatrix} -3-t & 0 & 0 & 1 \\ -16 & -t & 0 & 4 \\ -7 & 1 & 2-t & 1 \\ -11 & t-4 & \underset{= -(2-t)}{-2+t} & 5-t \end{bmatrix}$$

$$\underset{\substack{col_3 \text{ has} \\ \text{a factor } (t-2)}}{=} -(t-2) \det\begin{bmatrix} -3-t & 0 & 0 & 1 \\ -16 & -t & 0 & 4 \\ -7 & 1 & 1 & 1 \\ -11 & t-4 & -1 & 5-t \end{bmatrix} \underset{\substack{col_2 \to col_2 - col_3 \\ col_4 \to col_4 - col_3}}{=} (t-2) \det \begin{bmatrix} -3-t & 0 & 0 & 1 \\ -16 & -t & 0 & 4 \\ -7 & 0 & 1 & 0 \\ -11 & t-3 & -1 & 6-t \end{bmatrix}$$

$$\underset{col_1 \to col_1 + 7col_3}{=} (t-2) \det\begin{pmatrix} -3-t & 0 & 0 & 1 \\ -16 & -t & 0 & 4 \\ 0 & 0 & 1 & 0 \\ -18 & t-3 & -1 & 6-t \end{pmatrix} \underset{row_1 \leftrightarrow row_3}{=} +(t-2) \det\begin{pmatrix} 0 & 0 & 1 & 0 \\ -16 & -t & 0 & 4 \\ -3-t & 0 & 0 & 1 \\ -18 & t-3 & -1 & 6-t \end{pmatrix}$$

$$= (t-2)\,1\,(-1)^{1+3}\det\begin{bmatrix}-16 & -t & 4\\ -3-t & 0 & 1\\ -18 & t-3 & 6-t\end{bmatrix} = (t-2)\det\begin{bmatrix}-16 & -t & 4\\ -3-t & 0 & 1\\ -18 & t-3 & 6-t\end{bmatrix}$$

$$= (t-2)\Big(-16\big(-(t-3)\big) - (-t)\big((-3-t)(6-t)+18\big) + 4\big(-(t+3)(t-3)\big)\Big)$$

$$= (t-2)\big(16t-48+t(-18+3t-6t+t^2+18)-4(t^2-9)\big)$$

$$= (t-2)(16t-48-3t^2+t^3-4t^2+36) = (t-2)(t^3-7t^2+16t-12)$$

Note: $t^3-7t^2+16t-12$ has roots 3 & 2 (with mult $=2$ because it's derivative also has 2 as a root)

So $\boxed{P_A(t) = (t-2)^3(t-3)}$

• E_3:

$$A-3I_4 = \begin{bmatrix}-6 & 1 & 1 & 1\\ -16 & 1 & 4 & 4\\ -7 & 2 & 0 & 1\\ -11 & 1 & 3 & 2\end{bmatrix} \xrightarrow[R_3\to R_3-R_1]{} \begin{bmatrix}-6 & 1 & 1 & 1\\ -16 & 1 & 4 & 4\\ -1 & 1 & -1 & 0\\ -11 & 1 & 3 & 2\end{bmatrix} \xrightarrow[R_1\leftrightarrow R_3]{} \begin{bmatrix}-1 & 1 & -1 & 0\\ -16 & 1 & 4 & 4\\ -6 & 1 & 1 & 1\\ -11 & 1 & 3 & 2\end{bmatrix}$$

$$\xrightarrow[\substack{R_2\to R_2-16R_1\\ R_3\to R_3-6R_1\\ R_4\to R_4-11R_1}]{} \begin{bmatrix}-1 & 1 & -1 & 0\\ 0 & -15 & 20 & 4\\ 0 & -5 & 7 & 1\\ 0 & -10 & 14 & 2\end{bmatrix} \xrightarrow[\substack{R_2\to R_2-3R_3\\ R_4\to R_4-2R_2}]{} \begin{bmatrix}-1 & 1 & -1 & 0\\ 0 & 0 & -1 & 1\\ 0 & -5 & 7 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} \xrightarrow[R_2\leftrightarrow R_3]{} \begin{bmatrix}-1 & 1 & -1 & 0\\ 0 & -5 & 7 & 1\\ 0 & 0 & -1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

So $\begin{cases}-x_1+x_2-x_3=0 & \rightsquigarrow x_1 = x_2-x_3 = \frac{8}{5}x_4 - x_4 = \frac{3}{5}x_4\\ -5x_2+7x_3+x_4=0 & \rightsquigarrow 5x_2 = 8x_4 \rightsquigarrow x_2 = \frac{8}{5}x_4\\ -x_3+x_4=0 & \rightsquigarrow x_3 = x_4\end{cases}$

So $\underline{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}3/5\,x_4\\ 8/5\,x_4\\ x_4\\ x_4\end{bmatrix} = \frac{x_4}{5}\begin{bmatrix}3\\8\\5\\5\end{bmatrix}$ for any x_4, so $\boxed{E_3 = \mathrm{Span}\left(\begin{bmatrix}3\\8\\5\\5\end{bmatrix}\right)}$

• E_2:

$$A-2I_4 = \begin{bmatrix}-5 & 1 & 1 & 1\\ -16 & 2 & 4 & 4\\ -7 & 2 & 1 & 1\\ -11 & 1 & 3 & 3\end{bmatrix} \xrightarrow[R_2\to R_2-3R_1]{} \begin{bmatrix}-5 & 1 & 1 & 1\\ -1 & -1 & 1 & 1\\ -7 & 2 & 1 & 1\\ -11 & 1 & 3 & 3\end{bmatrix} \xrightarrow[R_2\leftrightarrow R_1]{} \begin{bmatrix}-1 & -1 & 1 & 1\\ -5 & 1 & 1 & 1\\ -7 & 2 & 1 & 1\\ -11 & 1 & 3 & 3\end{bmatrix}$$

$$\xrightarrow[\substack{R_2\to R_2-5R_1\\ R_3\to R_3-7R_1\\ R_4\to R_4-11R_1}]{} \begin{bmatrix}-1 & -1 & 1 & 1\\ 0 & 6 & -4 & -4\\ 0 & 9 & -6 & -6\\ 0 & 12 & -8 & -8\end{bmatrix} \xrightarrow[\substack{R_3\to R_3-\frac{3}{2}R_2\\ R_4\to R_4-2R_2}]{} \begin{bmatrix}-1 & -1 & 1 & 1\\ 0 & 6 & -4 & -4\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \xrightarrow[R_2\to \frac{R_2}{2}]{} \begin{bmatrix}-1 & -1 & 1 & 1\\ 0 & 3 & -2 & -2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

So $\begin{cases}-x_1-x_2+x_3+x_4=0 & \rightsquigarrow x_1 = -x_2+x_3+x_4 = \frac{1}{3}x_3+\frac{1}{3}x_4\\ 3x_2-2x_3-2x_4=0 & \rightsquigarrow x_2 = \frac{2}{3}x_3+\frac{2}{3}x_4\end{cases}$

$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}\frac{1}{3}x_3+\frac{1}{3}x_4\\ \frac{2}{3}x_3+\frac{2}{3}x_4\\ x_3\\ x_4\end{bmatrix} = \frac{x_3}{3}\begin{bmatrix}1\\2\\3\\0\end{bmatrix} + \frac{x_4}{3}\begin{bmatrix}1\\2\\0\\3\end{bmatrix}$ for any x_3, x_4: So $\boxed{E_2 = \mathrm{Span}\left(\begin{bmatrix}1\\2\\3\\0\end{bmatrix}, \begin{bmatrix}1\\2\\0\\3\end{bmatrix}\right)}$

<u>Conclusion</u>: $\lambda = 2$ eigenvalue with alg mult $= 3$ & geom mult $= 2$
$\lambda = 3$ eigenvalue with alg mult $= 1$ & geom mult $= 1$

So A is <u>defective</u> (because of $\lambda = 2$)

<u>Prop</u>: For any eigenvalue λ of an $n \times n$ matrix A, we have
alg mult $(\lambda) \geq$ geom. mult $(\lambda) \geq 1$.
So if alg mult $(\lambda) = 1$, we have alg mult (λ) = geom mult $(\lambda) = 1$.

<u>Theorem</u>: Given a matrix A of size $n \times n$ and a list of <u>distinct</u> eigenvalues $\lambda_1, \ldots, \lambda_k$, each with a nonzero eigenvector v_j, $j = 1, \ldots, k$ $[A v_j = \lambda_j v_j]$
Then $S = \{v_1, \ldots, v_k\}$ is a linearly independent set in $\mathbb{R}^n$.

<u>Proof</u>: • If $k = 1$, the result is true because a nonzero vector is always l.i.
• Assume $k > 1$ but the set S is linearly dependent. Then, we can find some $m = 2, \ldots, k$ where:
• $S_1 = \{v_1, \ldots, v_{m-1}\}$ is l.i.
• $S_2 = \{v_1, \ldots, v_{m-1}, v_m\}$ is l.d.

Write a non trivial linear combination showing that S_2 is l.d.:

$$(*) \qquad c_1 v_1 + c_2 v_2 + \cdots + c_{m-1} v_{m-1} + c_m v_m = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

but not all $c_1, \ldots, c_m$ are 0. Since the 1^{st} $m-1$ vectors are l.i, we know $c_m \neq 0$.

Multiply $(*)$ by A:

$$A(c_1 v_1 + \cdots + c_m v_m) = A \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$c_1 \underbrace{A v_1}_{\lambda_1 v_1} + \cdots + c_m \underbrace{A v_m}_{\lambda_m v_m} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$c_1 \lambda_1 v_1 + \cdots + c_{m-1} \lambda_{m-1} v_{m-1} + c_m \lambda_m v_m = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (**)$$

Performing $(**) - \lambda_m (*)$ gives:

$$c_1 \underbrace{(\lambda_1 - \lambda_m)}_{\neq 0} v_1 + \cdots + c_{m-1} \underbrace{(\lambda_{m-1} - \lambda_m)}_{\neq 0} v_{m-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since S_1 is l.i we get

$$\begin{aligned} c_1(\lambda_1-\lambda_m)=0 &\xrightarrow[\lambda_1\neq\lambda_m]{} c_1=0 \\ c_2(\lambda_2-\lambda_m)=0 &\xrightarrow[\lambda_2\neq\lambda_m]{} c_2=0 \\ &\vdots \\ c_{m-1}(\lambda_{m-1}-\lambda_m)=0 &\xrightarrow[\lambda_{m-1}\neq\lambda_m]{} c_{m-1}=0 \end{aligned}$$

Then, we go back to $(*)$ & get $\underline{0}+c_m v_m=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$ & $v_m\neq\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$, so $c_m=0$, contradicting our original statement.

We conclude the set S is l.i. $\square$

<u>Back to the example</u>: $S=\left\{\begin{bmatrix}3\\8\\5\\5\end{bmatrix},\begin{bmatrix}1\\2\\3\\0\end{bmatrix}\right\}$ is l.i by the Thm (and we can check it in the example!)

Also $S=\left\{\begin{bmatrix}3\\8\\5\\5\end{bmatrix},\begin{bmatrix}1\\2\\0\\3\end{bmatrix}\right\}$ ($\lambda=3$, $\lambda=2$) is l.i by the Theorem. (we can also check it!)

<u>Corollary</u>: If A of size $n\times n$ has n distinct eigenvalues, then A has a basis of eigenvectors (one per eigenvalue).