Lecture XXXIV: §4.6 Complex eigenvalues & eigenvectors

§1 Complex Numbers:

Motivation:

① **Algebraic**: We want to extend $\mathbb{R}$ so that every polynomial in one variable has a root (Eg: x^2+1 has no real roots)

② **geometric**: Is there a way to multiply points in $\mathbb{R}^2$?

We want to multiply 2 vectors $(\vec{u},\vec{v})$ in $\mathbb{R}^2$ & get a new one, so that:

$$\begin{cases} \vec{v}\cdot\vec{u} = \vec{u}\cdot\vec{v} & \text{(Commutative)} \\ \vec{v}\cdot(\vec{u}+\vec{w}) = \vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{w} & \text{(Distributive)} \end{cases}$$

and so that $\begin{bmatrix} a \\ 0 \end{bmatrix}\cdot\begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} ab \\ 0 \end{bmatrix}$

- In particular $\vec{v}\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{v}$ ($\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the unit for multiplication)
- Additionally: every $\vec{v} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ must be invertible: we must have $\vec{u}$, with $\vec{v}\cdot\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Definition: A complex number z is given by a pair of real numbers (a,b) written as $z = a+ib$. Call $\mathbb{C}$ the set of complex numbers.

$$\begin{cases} a = \text{Real part of } z = \operatorname{Re}(z) \\ b = \text{Imaginary part of } z = \operatorname{Im}(z). \end{cases}$$

- $\boxed{i \text{ satisfies } i^2 = -1.}$
- reals are complex numbers $a = a+i0$

Q: How do we add & multiply complex numbers?

(1) **Addition**: is done componentwise:

$$(a+ib)+(c+id) = (a+c)+i(b+d)$$

(2) **Multiplication**:

$$\begin{matrix} z = a+ib \\ w = c+id \end{matrix} \rightsquigarrow \boxed{zw = (ac-bd)+i(ad+bc)}$$

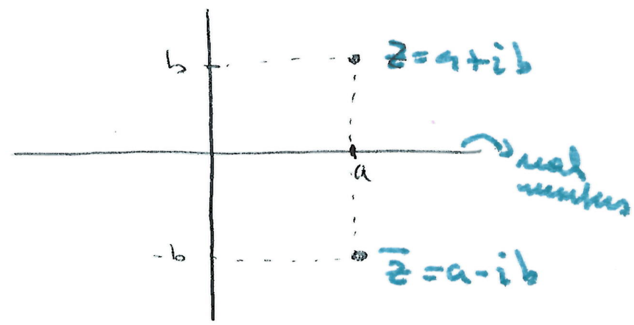
- Distribute Product
- Use $i^2=-1$.

Eg: $\begin{matrix} z = 1+i \\ w = 2+3i \end{matrix} \rightsquigarrow zw = (2-3)+(3+2)i = -1+5i$

$= 1(2+3i)+i(2+3i) = 2+3i+2i+3\underbrace{i^2}_{=-1}$

$= (2-3)+i(3+2) = -1+5i$

- As a vector $z = a+ib$ has magnitude $|z| = \sqrt{a^2+b^2}$ (We call it the modulus of z)
- Definition: Given $z = a+ib$, its complex conjugate is $\bar{z} = a - ib$.



Remark: In particular $z = \bar{z}$ if and only if $z = a + i0$, meaning z is a real number

Properties
(1) $\overline{z+w} = \bar{z} + \bar{w}$
(2) $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$
(3) $z\bar{z} = |z|^2$

As a consequence (if $z \neq 0$): $z\dfrac{\bar{z}}{|z|^2} = 1$, so $\boxed{z^{-1} = \dfrac{\bar{z}}{|z|^2}}$

We can use this to write any quotient $\dfrac{a+bi}{c+di}$ as a complex number:

$$\frac{a+bi}{c+di} = (a+bi)(c+di)^{-1} = (a+bi)\frac{(c-di)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

Examples (1) $z = 1+i \rightsquigarrow \bar{z} = 1-i$, $|z|^2 = 1^2+1^2 = 2$ so $z^{-1} = \dfrac{1-i}{2} = \dfrac{1}{2} - \dfrac{1}{2}i$

Then $\dfrac{2+i}{1+i} = (2+i)\left(\dfrac{1}{2} - \dfrac{1}{2}i\right) = \left(1+\dfrac{1}{2}\right) + \left(-1+\dfrac{1}{2}\right)i = \dfrac{3}{2} - \dfrac{1}{2}i$.

(2) $z = 1-3i$, $w = 2+4i$ $\rightsquigarrow z+w = 3+i$, $zw = (2+12)+(4-6)i = 14-2i$

$\bar{z} = 1+3i$, $\bar{w} = 2-4i$

$$\left.\begin{array}{l}\overline{z+w} = 3-i \\ \bar{z}+\bar{w} = 3+(3-4)i = 3-i\end{array}\right\} = \text{ as expected by Prop 1}$$

$$\left.\begin{array}{l}\bar{z}\cdot\bar{w} = (2+12)+(-4+6)i = 14+2i \\ \overline{zw} = \overline{(14-2i)} = 14+2i\end{array}\right\} = \text{ as expected by Prop 2}$$

Thm (Fundamental Theorem of Algebra) Every non constant polynomial in one variable with complex coefficients has a root in $\mathbb{C}$.

Example (quadratic polynomials) $P_{(x)} = ax^2+bx+c$. roots $= \dfrac{-b \pm \sqrt{b^2-4ac}}{2}$ in $\mathbb{C}$

b/c given complex numbers a, b, c, we always can find z, with $z^2 = b^2-4ac$.

Example: $b^2-4ac = -16$, then $z = 4i$ gives $z^2 = 4^2 i^2 = 16(-1) = -16$.

§2 Vectors in $\mathbb{C}^n$:

We write $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ but allow each coordinate to be in $\mathbb{C}$, rather than just real numbers

• Use conjugate to define magnitudes of complex vectors

Complex conjugate of v is $\bar{v} = \begin{bmatrix} \bar{v}_1 \\ \vdots \\ \bar{v}_n \end{bmatrix}$ Note: $\bar{v}_j v_j = |v_j|^2$ for $j=1,\dots,n$

Then $\|v\| = \sqrt{|v_1|^2 + \dots + |v_n|^2} = \sqrt{\bar{v}^T \cdot v}$

Notice that $\mathbb{R}^n \subseteq \mathbb{C}^n$ & the definition of magnitude in $\mathbb{C}^n$ agrees with the old definition in $\mathbb{R}^n$ because $\bar{z} = z$ if z is in $\mathbb{R}$.

Example $x = \begin{bmatrix} 2 \\ 1-i \\ 3+i \end{bmatrix}$ $y = \begin{bmatrix} i \\ 1+i \\ 2-i \end{bmatrix}$

$$x^T \cdot y = 2i + (1-i)(1+i) + (3+i)(2-i)$$
$$= 2i + (1+1) + i0 + (6+1) + i(-3+2) = 2i + 2 + 7 - i = \boxed{9+i}$$

usual dot product

$$\|x\|^2 = \bar{x}^T x = [2, 1+i, 3-i] \begin{bmatrix} 2 \\ 1-i \\ 3+i \end{bmatrix} = 4 + (1+1) + (9+1) = 4+2+10 = 16$$

so $\|x\| = \sqrt{16} = 4$.

§3 Eigenvalues in $\mathbb{C}$:

Example: $A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ $P_A(t) = \det\begin{pmatrix} 3-t & 1 \\ -2 & 1-t \end{pmatrix} = t^2 - 4t + 5$

Roots $\lambda = \dfrac{4 \pm \sqrt{-4}}{2} = \dfrac{4 \pm 2i}{2} = 2 \pm i$ $\nearrow 2+i$, $\searrow 2-i$