Lecture XXXV: §4.6 (cont) Complex eigenvectors & 4.7 Similarity

Recall: • Complex numbers: $z = a+bi$, a, b in $\mathbb{R}$, i satisfies $i^2=-1$
$a = Re(z)$
$b = Im(z)$
• Addition $(a+bi)+(c+di) = (a+c)+i(b+d)$
• Multiplication: $(a+bi)\cdot(c+di) = (ac-bd)+i(ad+bc)$
• Complex conjugation: $\overline{z} = a-bi$
• Norm: $|z|^2 = z\cdot\overline{z} = a^2+b^2$

$\Big\}\rightsquigarrow$ If $z\neq 0$: $z^{-1} = \dfrac{\overline{z}}{|z|^2}$

Thm: Every polynomial in $\mathbb{C}[x]$ of degree ≥ 1 has a root in $\mathbb{C}$

Q: What about polynomials in $\mathbb{R}[x]$?

2 types of roots → real roots
→ non-real complex roots: come in pairs $(\alpha, \overline{\alpha})$ ⇝ conjugate pairs

Why? If $f(\alpha)=0$ & f is in $\mathbb{R}[x]$ $\quad f(x) = a_0 + a_1 x + \cdots + a_d x^d$.

$$f(\alpha) = a_0 + a_1\alpha + \cdots + a_d \alpha^d = 0$$

We conjugate: $0 = \overline{f(\alpha)} = \overline{a_0 + a_1\alpha + \cdots + a_d\alpha^d} = \overline{a_0} + \overline{a_1}\,\overline{\alpha} + \cdots + \overline{a_d}(\overline{\alpha})^d$

but $a_0, \ldots, a_d$ are real, so $= a_0 + a_1\overline{\alpha} + \cdots + a_d(\overline{\alpha})^d = f(\overline{\alpha})$

So if α is a root, then $\overline{\alpha}$ is also a root!

Example: $f(x) = x^3 - x^2 + x - 1 = (x-1)(x^2+1) = (x-1)\underbrace{(x-i)(x+i)}$ complex conjugate

• **Application**: If $P_A(t)$ is the characteristic polynomial of an $n\times n$ matrix A with real entries, the eigenvalues of A are the roots of P_A, so they are real or come in conjugate pairs..

§1 Eigenvectors in $\mathbb{C}^n$:

If λ is a non-real eigenvalue of A, then $E_\lambda = \{v : Av = \lambda v\}$ must consist of vectors in $\mathbb{C}^n$ because if $v\in\mathbb{R}^n$, then Av ~~is~~ in $\mathbb{R}^n$ (so λ would be in $\mathbb{R}$).

We operate as usual, solving $(A-\lambda I_n)\begin{bmatrix} x_1 \\ \vdots \\ x_n\end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0\end{bmatrix}$ over $\mathbb{C}$.

Example: $A = \begin{bmatrix} 3 & 1 \\ -2 & 1\end{bmatrix}$ $\quad P_A(t) = \det\begin{pmatrix} 3-t & 1 \\ -2 & 1-t\end{pmatrix} = t^2 - 4t + 5$

Roots: $\lambda = \dfrac{4 \pm \sqrt{(-4)^2 - 4\cdot 5}}{2} = \dfrac{4\pm\sqrt{-4}}{2} = \dfrac{4\pm 2\sqrt{-1}}{2} = 2 \pm i$ (conjugate pairs)

2 Eigenspaces → $E_{2+i} = \text{Nullspace}\left(A-(2+i)I_2\right) = \text{Nullspace}\begin{pmatrix}1-i & 1\\ -2 & -1-i\end{pmatrix}$

→ $E_{2-i} = \text{Nullspace}\left(A-(2-i)I_2\right) = \text{NullSpace}\begin{pmatrix}1+i & 1\\ -2 & -1+i\end{pmatrix}$

Solve using Gauss-Jordan elimination (can use $\mathbb{C}$-numbers as scalars for our operations!)

• $\underline{E_{2+i}}$: $\begin{bmatrix}1-i & 1\\ -2 & -1-i\end{bmatrix} \xrightarrow[R_1\to(1+i)R_1]{} \begin{bmatrix}2 & 1+i\\ -2 & -1-i\end{bmatrix} \xrightarrow[R_2\to R_2-R_1]{} \begin{bmatrix}2 & 1+i\\ 0 & 0\end{bmatrix}$

So $2x_1 + (1+i)x_2 = 0$, so $x_1 = \frac{-1-i}{2}x_2$

$\underline{x} = \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}\frac{-1-i}{2}x_2\\ x_2\end{bmatrix} = \frac{x_2}{2}\begin{bmatrix}-1-i\\ 2\end{bmatrix}$ ⇝ $E_{2+i} = \text{Span}\left(\begin{bmatrix}-1-i\\ 2\end{bmatrix}\right)$

• Similarly: $E_{2-i} = \text{Span}\left(\begin{bmatrix}-1+i\\ 2\end{bmatrix}\right)$

<u>Check</u>: $A\begin{bmatrix}-1-i\\ 2\end{bmatrix} = \begin{bmatrix}3 & 1\\ -2 & 1\end{bmatrix}\begin{bmatrix}-1-i\\ 2\end{bmatrix} = \begin{bmatrix}(-3+2)-3i\\ (+2+2)+2i\end{bmatrix} = \begin{bmatrix}-1-3i\\ 4+2i\end{bmatrix}$

& $(2+i)\begin{bmatrix}-1-i\\ 2\end{bmatrix} = \begin{bmatrix}-(2+i)(1+i)\\ 4+2i\end{bmatrix} = \begin{bmatrix}-((2-1)+(2+1)i)\\ 4+2i\end{bmatrix} = \begin{bmatrix}-1-3i\\ 4+2i\end{bmatrix}$ ✓

$A\begin{bmatrix}-1+i\\ 2\end{bmatrix} = \begin{bmatrix}3 & 1\\ -2 & 1\end{bmatrix}\begin{bmatrix}-1+i\\ 2\end{bmatrix} = \begin{bmatrix}(-3+2)+3i\\ (2+2)+(-2)i\end{bmatrix} = \begin{bmatrix}-1+3i\\ 4-2i\end{bmatrix}$

& $(2-i)\begin{bmatrix}-1+i\\ 2\end{bmatrix} = \begin{bmatrix}(-2+1)+i(2+1)\\ 4-2i\end{bmatrix} = \begin{bmatrix}-1+3i\\ 4-2i\end{bmatrix}$ ✓

<u>Notice</u>: If λ is an eigenvalue of A, non real, with an eigenvector $v = \begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$ in $\mathbb{C}^n$, then $\bar{\lambda}$ is an eigenvalue of A & $\bar{v} = \begin{bmatrix}\bar{x}_1\\ \vdots\\ \bar{x}_n\end{bmatrix}$ in $\mathbb{C}^n$ is an eigenvector.

<u>Proposition</u>: if λ is a non real eigenvalue of A & $B = \{v_1, \dots v_n\}$ is a basis of E_λ (vectors in $\mathbb{C}^n$), then $\tilde{B} = \{\bar{v}_1, \dots, \bar{v}_n\}$ is a basis of $E_{\bar{\lambda}}$.

<u>Why?</u> $Av = \lambda v$ & conjugate $\overline{Av} = \overline{\lambda v}$.

$v \neq \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$ $\qquad A\bar{v} = \bar{A}\bar{v} = \bar{\lambda}\bar{v}$

A has real entries

So $\bar{v}$ is a nontrivial solution to $A\bar{v} = \bar{\lambda}\bar{v}$ & so $\bar{v}$ in $E_{\bar{\lambda}}$.

• Linear independence is immediate.

<u>Key Thm</u>: If A is an $(n\times n)$ real <u>symmetric</u> matrix, then all the eigenvalues of A are REAL. (Next Time: we have a <u>basis</u> of eigenvectors)

Proof: Assume λ is an eigenvalue of A with eigenvector $v \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, v in $\mathbb{C}^n$ (λ in $\mathbb{C}$)

Then $\bar{v}^T A v = \bar{v}^T(\lambda v) = \lambda(\bar{v}^T v) = \lambda \|v\|^2$ (*) $\left(\bar{v}^T v = [\bar{x}_1, \dots \bar{x}_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = |x_1|^2 + \dots + |x_n|^2\right)$

But $\bar{v}^T(\lambda v) = (\lambda v)^T \bar{v} = (Av)^T \bar{v} = (v^T A^T)\bar{v} \underset{\substack{\downarrow \\ \text{Asymmetric}}}{=} v^T A \bar{v} \underset{\substack{\downarrow \\ \text{Prop} \\ \bar{v} \text{ in } E_{\bar{\lambda}}}}{=} v^T(\bar{\lambda}\bar{v})$

$= v^T \bar{\lambda} \bar{v} = \bar{\lambda} v^T \bar{v} = \bar{\lambda}\|v\|^2$ (**)

We conclude $\lambda \|v\|^2 = \bar{v}^T A v = \bar{\lambda}\|v\|^2$ & $\|v\| \neq 0$ so $\boxed{\lambda = \bar{\lambda}}$,

so λ is a real number □.

Example $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad P_A(t) = \det\begin{bmatrix} -t & 1 \\ 1 & -t \end{bmatrix} = t^2 - 1 = (t+1)(t-1) \rightsquigarrow A$ has

2 distinct eigenvalues, each with 1 eigenvector $E_1 = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$, $E_{-1} = \operatorname{Span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

- Basis of eigenvectors: $\mathcal{B}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ & $T: \mathbb{R}^2 \to \mathbb{R}^2$ $Tv = Av$ satisfies $[T]_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

§2 Similarity:

Definition: Two matrices A, C of size $(n \times n)$ are similar if there is a nonsingular $(n \times n)$ matrix S such that $C = S^{-1}AS$.

Motivation: A & C will be matrices of a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ with respect to 2 basis. ($T(v) = Av$ so $A = [T]_{EE}$, If $C = [T]_{\mathcal{B}}$ $\mathcal{B} = \{v_1, \dots v_n\}$ then $S = [v_1 \cdots v_n]$)

Proposition: Similar matrices have the same characteristic polynomial.

Proof: $P_C(t) = \det(S^{-1}AS - tI_n) = \det(S^{-1}(A - tI_n)S)$

$\underset{\substack{\uparrow \\ \det \text{ is multiplicative}}}{=} \det(S^{-1})\det(A - tI_n)\det S = \frac{1}{\det S} P_A(t) \det S = P_A(t)$

Remark: The converse fails!

Example: $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ & $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ both have $P_{A}(t) = (1-t)^2$

But $S I_2 S^{-1} = I_2$ always so $I_2 \neq S^{-1}AS$.

Proposition: If v is an eigenvector of C with eigenvalue λ, then $w = Sv$ is an eigenvector of A with eigenvalue λ.

Proof: $\lambda v = Cv = S^{-1}ASv$ so $A(Sv) = S\lambda v = \lambda(Sv)$.

Example $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ → basis of eigenvectors

$$S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad S^{-1} = \frac{1}{\det S} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$\det S = -2$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \stackrel{?}{=} S^{-1} A S = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \checkmark$$

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $T_{(v)} = Av$. $[T]_{EE} = A$ & $[T]_B = C$.

Next Time: generalize this diagonalization procedure (whenever possible!).