Lecture XXXVI: §4.7 Similarity & Diagonalization (cont.)

Recall: Two matrices A, C real of size $n\times n$ are similar if we can find S invertible $n\times n$ matrix with $C = S^{-1}AS$.

Test: If A & C are similar, then $P_A(t) = P_C(t)$

But the converse is not true!

Example: $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ & $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ both have $P_A(t) = P_{I_2}(t) = (1-t)^2$

But if A & I_2 are similar, then $A = \underbrace{S I_2 S^{-1}}_{= I_2}$ for some S invertible

So this can't happen

(I_2 is only similar to itself!)

Proposition: If v is an eigenvector of $C = S^{-1}AS$ with eigenvalue λ, then $w = Sv$ is an eigenvector of A with eigenvalue λ.

Proof $\lambda v = Cv = S^{-1}ASv$ so $S(\lambda v) = SS^{-1}ASv = A(Sv)$

So $Aw = \lambda w$ & w is an eigenvector of A with eigenvalue λ.

$\lambda(Sv)$

§1. Diagonalization:

Def: We say A ($n\times n$ real matrix) is diagonalizable if it is similar to a diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ 0 & \cdots & 0 & d_n \end{pmatrix}$$

Note $P_A(t) = P_D(t) = (d_1 - t)\cdots(d_n - t)$ so the diagonal entries of A are the eigenvalues of A (with the corresponding algebraic multiplicity)

↳ order of d_i as a root of $P_A(t)$

Q: Why is this useful?

If $D = S^{-1}AS$, then $D^k = (S^{-1}AS)^k = \underbrace{(S^{-1}A\cancel{S})(\cancel{S^{-1}}A\cancel{S})\cdots(\cancel{S^{-1}}AS)}_{k \text{ times}}$

$= S^{-1}A^kS$

So $A^k = SD^kS^{-1}$

easy to do! $D^k = \begin{pmatrix} d_1^k & & 0 \\ & \ddots & \\ 0 & & d_n^k \end{pmatrix}$

Thm: An ($n\times n$) matrix is diagonalizable if and only if A has a set of n linearly independent eigenvectors (they are a basis of $\mathbb{C}^n$).

Note: the eigenvectors need not be real, need to work with complex vectors if we have complex eigenvalues.

Proof ($\Leftarrow$) Assume $B=\{v_1, \dots v_n\}$ is a basis of eigenvectors of A, write $S=[v_1 \cdots v_n]$. Since the columns of S are l.i, then S is invertible.

Write $Av_i = \lambda_i v_i$ for $i=1,\dots,n$ (λ_i is the eigenvalue for A).

Then
$$AS = A[v_1 \cdots v_n] \underset{\substack{\downarrow \\ \text{each col is } Av_i = \lambda_i v_i.}}{=} [\lambda_1 v_1 \cdots \lambda_n v_n]$$

So
$$S^{-1}AS = S^{-1}[\lambda_1 v_1 \cdots \lambda_n v_n]$$
$$= [\lambda_1 \underbrace{S^{-1}v_1}_{\substack{\shortparallel \\ S^{-1}\text{col}_1 S = \text{col}_1(S^{-1}S) = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}}} \cdots \lambda_n \underbrace{S^{-1}v_n}_{S^{-1}\text{col}_n S = \text{col}_n(S^{-1}S) = \begin{bmatrix}0\\\vdots\\1\end{bmatrix}}] = \begin{bmatrix}\lambda_1 & & 0\\ 0 & \ddots & \vdots \\ 0 & & \lambda_n\end{bmatrix} = \text{Diag}(\lambda_1, \dots, \lambda_n)$$

so A is diagonalizable.

($\Rightarrow$) Conversely, assume A is diagonalizable & write $D = C^{-1}AC$ for some C invertible.

Then $CD = AC$.

Write column vectors of $C = [c_1 \cdots c_n]$ each $c_1, \dots, c_n$ in $\mathbb{K}^n$

Write $D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n\end{bmatrix}$

Then
$$CD = C[d_1 e_1 \cdots d_n e_n] = [Cd_1 e_1 \cdots Cd_n e_n]$$
$$= [d_1 C_1 \cdots d_n C_n] \quad \text{because } Ce_i = \text{col}_i C \text{ for } i=1,\dots,n$$

Now $AC = [AC_1 \cdots AC_n]$

Then $AC = CD$ translates to $AC_1 = d_1 C_1, \dots, AC_n = d_n C_n$

So $\{C_1 \dots C_n\}$ is a set of eigenvectors of A (with eigenvalues $d_1 \dots d_n$) & they are l.i because they are the columns of ~~the~~ invertible matrix C. $\square$

<u>Remark</u>: The linearly indep eigenvectors are the columns of S for $D = S^{-1}AS$ & the diagonal matrix D has the corresponding eigenvalues (in the <u>same order</u>!)

<u>Example</u>: $A = \begin{bmatrix} 5 & -6 \\ 3 & -4\end{bmatrix}$ $\quad P_A(t) = (5-t)(-4-t)+18 = t^2 - t + 2 = (t-2)(t+1)$

A has 2 distinct eigenvalues, so we have a <u>basis</u> of eigenvectors. By the Thm A is diagonalizable (over $\mathbb{R}$ in this case!)

E_{-1}: Solve $\begin{bmatrix} 6 & -6 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow x_1 - x_2 = 0$, $\boxed{x_1 = x_2}$ So $E_{-1} = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

E_2: Solve $\begin{bmatrix} 3 & -6 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow x_1 - 2x_2 = 0$, $\boxed{x_1 = 2x_2}$ So $E_2 = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$

$\rightsquigarrow S = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ & $S^{-1} = \dfrac{1}{\det S}\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$

eigenvalue $= -1$ eigenvalue $= 2$

$\det S = 1 - 2 = -1$

So $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \overset{?}{=} S^{-1}AS = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \checkmark$

Remark: Sufficient condition to be diagonalizable: A has exactly n distinct eigenvalues.

Example 2: Calculate A^{10}.

$$A^{15} = S D^{10} S^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1024 & -1024 \end{bmatrix}$$
$$= \begin{bmatrix} 2047 & -2046 \\ 1023 & -1022 \end{bmatrix}$$

Example 3 (last time) $A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ eigenvalues: $2+i$, $2-i$

$E_{2+i} = \text{Span}\left(\begin{bmatrix} -1-i \\ 2 \end{bmatrix}\right)$, $E_{2-i} = \text{Span}\left(\begin{bmatrix} \overline{-1-i} \\ \overline{2} \end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix} -1+i \\ 2 \end{bmatrix}\right)$

$S = \begin{bmatrix} -1-i & -1+i \\ 2 & 2 \end{bmatrix}$ $\det S = 2(-1-i) - 2(-1+i) = -2+2-2i-2i = -4i$

$S^{-1} = \dfrac{1}{-4i}\begin{bmatrix} 2 & 1-i \\ -2 & -1-i \end{bmatrix} = \dfrac{-i}{-4}\begin{bmatrix} 2 & 1-i \\ -2 & -1-i \end{bmatrix} = \begin{bmatrix} i/2 & (1+i)/4 \\ -i/2 & (1-i)/4 \end{bmatrix}$

$i^{-1} = -i$

$D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix} = S^{-1}AS = \dfrac{1}{4}\begin{bmatrix} 2i & 1+i \\ -2i & 1-i \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -1-i & -1+i \\ 2 & 2 \end{bmatrix}$ $\checkmark$

§2 Real Symmetric matrices

Last time: All eigenvalues of A $n \times n$ real symmetric matrix are REAL.

Stronger statement: A is diagonalizable with eigenvectors in $\mathbb{R}^n$.

Furthermore, these matrices have a special structure:

Thm: Fix A a real $n \times n$ matrix.

orthogonal matrix

(1) If A is symmetric, then we can find Q an $n \times n$ matrix with $Q^T = Q^{-1}$ & a diagonal matrix D with $D = Q^T A Q$

(2) If $Q^T A Q = D$ with $Q^{-1} = Q^T$ & D diagonal, then A is symmetric.

- $Q^T = Q^{-1}$ means the columns of Q are an orthonormal basis for $\mathbb{R}^n$.

<u>Example</u>: $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $\quad P_A(t) = (1-t)^2 - 1 = (1-t+1)(1-t-1) = +t(t-2)$

$E_0 = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ $\quad E_2 = \text{NullSpace}\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

not orthonormal, but $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ is

Pick orthonormal vector: $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ $\quad Q^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = Q^{-1}$. because $v_1 \cdot v_1 = 1$, $v_2 \cdot v_2 = 1$, $v_1 \cdot v_2 = 0$.

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = Q^T A Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$