SOLUTIONS

Midterm 1
Math 2568 - Linear Algebra (Section 75)

Prof. Cueto

Friday Feb. 3rd 2017

- The use of class notes, book, formulae sheet or calculator is **not permitted**.
- In order to get full credit, you **must**:
  a) get the **correct answer**, and
  b) **show all your work** and/or explain the reasoning that leads to that answer.
- Answer the questions **in the spaces provided** on the question sheets. If you run out of room for an answer, continue on the back of the page.
- Please make sure the solutions you hand in are **legible and lucid**.
- You have **fifty-five minutes** to complete the exam.
- Do not forget to write your full name (in PRINT) in the space provided below and on the bottom of the last page.

Full Name (Print): ______________________________

Good luck!

Exercise 1. **[12 points]** Consider the systems of linear equations (with parameter a):

$$\begin{cases} x_1 + x_2 = 2 \\ x_1 - x_2 = 0 \\ 2x_1 - x_2 + (a^2 - a - 2)\, x_3 = a^2 + 2a - 7 \end{cases}$$

a) **[4 points]** For what values of a does the system have *infinitely many* solutions?

From eqns (1) & (2) : $x_1 = 1$, $x_2 = 1$ Substitute in 3rd

$2 - 1 + (a^2 - a - 2)\, x_3 = a^2 + 2a - 7$

$\Rightarrow \boxed{(a^2 - a - 2)\, x_3 = a^2 + 2a - 8}$

Factor : $\begin{cases} a^2 - a - 2 = (a+1)(a-2) \\ a^2 + 2a - 8 = (a+4)(a-2) \end{cases}$

If $\boxed{a = 2}$, the eqn becomes $0 \cdot x_3 = 0$ so infinitely many solutions

And it is the only case when this happens.

b) **[4 points]** For what values of a does the system have *no solutions*?

If $\boxed{a = -1}$, the eqn becomes $0 \cdot x_3 = -9$, so no soln

For any other value of a the system has a solution

c) **[4 points]** For what values of a do we have a solution ~~with~~ satisfying $x_3 = 0$? Is this the only solution with this property?

If $a \neq 2, -1$, the system has a unique solution

$x_1 = x_2 = 1$, $x_3 = \dfrac{a+4}{a+1}$

This value becomes 0 when $\boxed{a = -4}$

If $\boxed{a = 2}$, then x_3 can be anything but there is only one solution with $x_3 = 0$ $(x_1 = x_2 = 1)$

Answer = $\boxed{a = 2, -4}$

In both cases (1,1,0) it is the only solution with this property.

Exercise 2. **[12 points]**

Consider the matrix $A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \\ -3 & 1 & -3 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

a) **[6 points]** Determine conditions on $\mathbf{b}$ for which the system $A\mathbf{x} = \mathbf{b}$ has no solutions.

We use Gauss-Jordan:

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & b_1 \\ 2 & -1 & 1 & b_2 \\ -3 & 1 & -3 & b_3 \end{array}\right] \xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & b_1 \\ 0 & 1 & 3 & b_2 - 2b_1 \\ 0 & -2 & -6 & b_3 + 3b_1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + 2R_2]{} \left[\begin{array}{ccc|c} 1 & -1 & -1 & b_1 \\ 0 & 1 & 3 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 + 2(b_2 - 2b_1) \end{array}\right]$$

So the system is <u>inconsistent</u> <u>if and only if</u>

$$b_3 + 3b_1 + 2b_2 - 4b_1 = \boxed{-b_1 + 2b_2 + b_3 \neq 0}$$

b) **[3 points]** Write the vector form of the general solution of the system $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

We use (a): $[A \mid \vdots] \sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right]$ so we know the system has no solution.

c) **[3 points]** Are the columns of A linearly independent? Justify your answer.

Since we found a value of $\underline{b}$ giving an inconsistent system, we know A cannot be invertible, in particular it is singular & its columns are <u>linearly dependent</u>

Exercise 3. [10 points]

a) [**3 points**] Assume P, Q and R are nonsingular square matrices with $PQR = I_n$. Express Q^{-1} in terms of P and R.

They are invertible so by the algebraic properties of matrix multiplication we get $P^{-1}(PQR)R^{-1} = P^{-1} I_n R^{-1}$

$$Q = P^{-1}R^{-1}$$

Invert: $Q^{-1} = (P^{-1}R^{-1})^{-1} = \boxed{RP}$.

b) [**4 points**] Let $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. Calculate ABA, BAB and $-2((AB)^3)^T$.

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$ABA = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$BAB = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$(AB)^3 = (AB\,A)(B\,AB) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2$$

So $-2((AB)^3)^T = -2(-I_2)^T = 2I_2 = \boxed{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}}$

c) [**3 points**] Let $A=\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 4 \\ 0 & 2 & 9 \end{bmatrix}$ and $\mathbf{u}=\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$. Compute $\|A^{-1}\mathbf{u}\|$.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 2 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_2]{} \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 \to R_2 - 4R_3 \\ R_1 \to R_1 - 5R_3}]{} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 10 & -5 \\ 0 & 1 & 0 & 0 & 9 & -4 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right] \xrightarrow[R_1 \to R_1 - 3R_2]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -17 & 7 \\ 0 & 1 & 0 & 0 & 9 & -4 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$$

" A^{-1} "

$$A^{-1}\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$$

so $\|A^{-1}u\| = \sqrt{4+16+1} = \boxed{\sqrt{21}}$

Exercise 4. [12 points]

a) **[6 points]** Let $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Compute A^{25}, A^{26} and $(\frac{1}{2}A^{-1})^{40}$.

Do a few small cases: $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \rightsquigarrow A = A^{-1}$

$A^3 = A$

in general $A^{2n} = I_3 \quad A^{2n+1} = A$, so $\boxed{\begin{aligned} A^{25} &= A \\ A^{26} &= I_3 \end{aligned}}$

$$\left(\frac{1}{2}A^{-1}\right)^{40} = \left(\frac{1}{2}A\right)^{40} = \frac{1}{2^{40}}A^{40} = \frac{1}{2^{40}}I_3 = \begin{bmatrix} 1/2^{40} & 0 & 0 \\ 0 & 1/2^{40} & 0 \\ 0 & 0 & 1/2^{40} \end{bmatrix}$$

b) **[6 points]** A matrix A is called *idempotent* if $A^2 = A$. Show that if an idempotent matrix A is nonsingular then A must be the Identity matrix.

$A^2 = A$ so $A^2 - A = A(A - I_n) = 0$

If A is invertible, then multiply by A^{-1}.

$A^{-1}A(A - I_n) = A^{-1}0$

$I_n(A - I_n) = 0$

$A - I_n = 0$

$\boxed{A = I_n}$

Exercise 5. [**14 points**]

Determine whether each of the statements is True or False. **No justification is necessary.**

a) *True* *False* If an $m \times n$ system is inconsistent, then the reduced echelon form of the augmented matrix must have $n+1$ nonzero rows.

b) *True* *False* If a square matrix A can be reduced to the identity matrix by elementary row operations, then A is invertible.

c) *True* *False* A linear system with less equations than variables cannot have a unique solution.

d) *True* *False* Two linear systems are equivalent if they have the same solutions set.

e) *True* *False* Applying elementary row operations on an augmented matrix can change the solutions set of the associated linear system.

f) *True* *False* A homogeneous system of equations can be inconsistent.

g) *True* *False* A consistent 3×2 system of linear equations can have exactly two solutions.

For Grader's use only:

1			2			3			4		5							TOTAL
a	b	c	a	b	c	a	b	c	a	b	a	b	c	d	e	f	g	