SOLUTIONS

Midterm 2
Math 2568 - Linear Algebra (Section 75)

Prof. Cueto

Friday March 10th 2017

- The use of class notes, book, formulae sheet or calculator is **not permitted**.
- In order to get full credit, you **must**:
  a) get the **correct answer**, and
  b) **show all your work** and/or explain the reasoning that leads to that answer.
- Answer the questions **in the spaces provided** on the question sheets. If you run out of room for an answer, continue on the back of the page.
- You will find a sheet of **scratch paper** at the end of the exam. Feel free to remove it and use it at your convenience.
- Please make sure the solutions you hand in are **legible and lucid.**
- The exam is worth a total of **35 points**.
- You have **fifty-five minutes** to complete the exam.

Full Name (Print): ____________________

Good luck and have a fun Spring Break!

Exercise 1. **[7 points]** Consider the matrix $A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -2 & 5 & 4 \\ 1 & -1 & 0 & -7 \end{bmatrix}$.

a) **[2 points]** Show that the rows of A are linearly independent vectors.

Use G-J: $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -2 & 5 & 4 \\ 1 & -1 & 0 & -7 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & -10 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -14 \end{bmatrix}$

we get all nonzero rows, so they are l.i

Soln 2: $\begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 2 & 5 & 0 \\ 3 & 4 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ Want to show $a = b = c = 0$

$\begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 2 & 5 & 0 \\ 3 & 4 & -7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & -10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -14 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ RREF

Conclude $a = b = c = 0$ so the rows are l.i by definition

b) **[3 points]** Find a basis for the Null Space of A.

Use item (a) $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -2 & 5 & 4 \\ 1 & -1 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -14 \end{bmatrix} \xrightarrow{R_3/-14} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\uparrow$ dependent

$x_1 = x_2$
$x_3 = 0$
$x_4 = 0$

so $\underline{x} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$N(A)$ has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Check $A \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0$ & by rank-nullity Theorem: $4 = \underset{\text{rank}}{3} + \text{nullity}(A)$ so $\dim N(A) = 1$ ✓

c) **[2 points]** Show that the matrix A has rank 3.

$\text{rank}(A) = \text{rank}(A^T) = 3$ (3 columns of A^T are l.i by calculations in (a))

Alternative solution: use rank-nullity Theorem: $\text{rank}(A) = 4 - 1 = 3$

Exercise 2. [**8 points**] Fix $\mathbf{w_1} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{w_2} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{w_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

a) [**4 points**] Find the matrix of a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ where

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \mathbf{w_1}\,,\ T\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right) = \mathbf{w_2}\,,\ \text{and}\ T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \mathbf{w_3}.$$

We need to find expressions for $e_1, e_2, \& e_3$ in terms of the 3 vectors in $\mathbb{R}^3$

$$e_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \rightsquigarrow T(e_2) = T(e_1) - T\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right) = w_1 - w_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$e_3 = -2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightsquigarrow T(e_3) = -2T(e_1) + T\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$
$$= -2w_1 + w_2 + w_3$$

$$-2w_1 + w_2 + w_3 = -2\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Conclusion: The matrix is $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

b) [**4 points**] Using Gram-Schmidt find an **orthogonal basis** for the span of $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$.

$u_1 = w_1 \qquad u_1^T w_2 = 0 + 1 + 6 + 1 = 8, \qquad u_1^T w_3 = 3, \qquad \|u_1\|^2 = 6$

$$u_2 = w_2 - \frac{u_1^T w_2}{\|u_1\|^2} u_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} - \frac{8}{6}\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1-4/3 \\ 3-8/3 \\ 1-4/3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/3 \\ 1/3 \\ -1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

so $\|u_2\|^2 = \frac{1}{9} 3$ & $u_2^T \cdot w_3 = \frac{1}{3}[0, -1, 1, -1]\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = 0$

$$u_3 = w_3 - \frac{u_1^T w_3}{\|u_1\|^2} u_1 - \frac{u_2^T w_3}{\|u_2\|^2} u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{6}\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1-1/2 \\ 1-1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 0 \\ -1/2 \end{bmatrix}$$

Check $u_1 u_2 = \frac{1}{3}(-1+2-1) = 0$

$u_1 u_3 = 0 + \frac{1}{2} - \frac{1}{2} = 0$

$u_2 u_3 = -\frac{1}{2} + \frac{1}{2} = 0$

Answer: $\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

Exercise 3. **[6 points]** Consider the space $\mathcal{P}_2$ of polynomials $P(x)$ of degree at most 2.

a) **[4 points]** Show that $B = \{1 + x + x^2, -1 + x, 1 + 2x + x^2\}$ is a basis of $\mathcal{P}_2$.

Use coordinates with respect to the standard basis for $\mathcal{P}_2$, namely: $B' = \{1, x, x^2\}$

$[P_1]_{B'} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $[P_2]_{B'} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $[P_3]_{B'} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ coordinate in $\mathbb{R}^3$

So: we need to check if they are l.i (since dim $\mathbb{R}^3 = 3$, the coordinates will be a basis for $\mathbb{R}^3$, so B will be a basis for $\mathcal{P}_2$.

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

REF

Since we get all nonzero rows, the original vectors (rows) are l.i so the same happens to the 3 polynomials.

b) **[2 points]** Compute $[2\,x^2]_B$, that is the coordinates of $2\,x^2$ with respect to the basis B from item (a).

$$2x^2 \stackrel{?}{=} a(1 + x + x^2) + b(-1 + x) + c(1 + 2x + x^2)$$
$$= (a - b + c) + (a + b + 2c)x + (a + c)x^2$$

So $\begin{cases} a - b + c = 0 \\ a + b + 2c = 0 \\ a + c = 2 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -4 \\ 0 & 1 & 0 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & -4 \\ 0 & 1 & 0 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

$a = 6$
$b = 2$
$c = -4$

Check: $6 - 2 - 4 = 0$ ✓
$6 + 2 - 8 = 0$ ✓
$6 - 4 = 2$ ✓

Conclusion: $[2x^2]_B = \begin{bmatrix} 6 \\ 2 \\ -4 \end{bmatrix}$.

Exercise 4. **[6 points]** Consider the planes $\pi_1: \; 2x - y + 3z = 1$ and $\pi_2: \; x + 5y + z = 3$.

a) **[2 points]** Show that the two planes are perpendicular to each other, i.e., their normal vectors are perpendicular.

$\eta_1 = [2 \; -1 \; 3]^T$

$\eta_2 = \begin{bmatrix}1\\5\\1\end{bmatrix}$

so $\eta_1 \cdot \eta_2 = [2 \; -1 \; 3]\begin{bmatrix}1\\5\\1\end{bmatrix} = 2-5+3 = 0$

so $\eta_1 \perp \eta_2$ & so are the planes!

b) **[2 points]** Find the direction vector of the line where the two planes intersect.

We find the line by solving the equation.

$$\begin{cases} 2x - y + 3z = 1 \\ x + 5y + z = 3 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 1 & 5 & 1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & 1 & 3 \\ 2 & -1 & 3 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & 1 & 3 \\ 0 & -11 & 1 & -5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & 1 & 3 \\ 0 & 1 & -1/11 & 5/11 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 16/11 & 8/11 \\ 0 & 1 & -1/11 & 5/11 \end{array}\right]$$

$33 - 25 = 8$

so $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}8/11\\5/11\\0\end{bmatrix} + z\underbrace{\begin{bmatrix}-16/11\\+1/11\\1\end{bmatrix}}_{\text{direction vector}}$

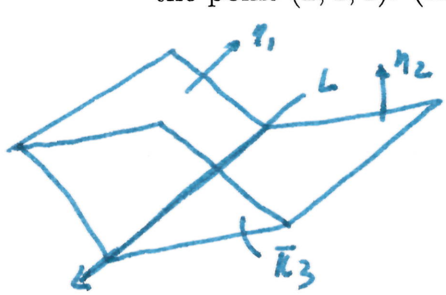
We take a multiple $= \begin{bmatrix}-16\\1\\11\end{bmatrix}$

Alternative

direction $= \eta_1 \times \eta_2$

$\begin{vmatrix} i & j & k \\ 2 & -1 & 3 \\ 1 & 5 & 1 \end{vmatrix} = -i\,16 - j(-1) + k(11) = \begin{bmatrix}-16\\1\\11\end{bmatrix}$

c) **[2 points]** Find a plane π_3 that is perpendicular to both π_1 an π_2, passing through the point $(2, 1, 1)$. (*Hint:* Use the previous item and draw a picture).



The direction of L is the normal direction to η_3 & point $= (2,1,1)$

$-16x + y + 11z = [-16 \; 1 \; 11]\begin{bmatrix}2\\1\\1\end{bmatrix} = -32+1+11 = -20$

So $\boxed{-16x + y + z = -20}$

normal $\eta_3 \perp \eta_1$

$\eta_3 \perp \eta_2$

Exercise 5. [8 points]

Determine whether each of the statements is True or False. **No justification is necessary.**

a) *True* *False* The space of 2×3 matrices has dimension 7.

b) *True* *False* If 2 matrices A, B of size $2{\times}2$ are linear indep., so are A and $A{+}B$.

c) *True* *False* A set of vectors spanning $\mathbb{R}^4$ has at least 4 vectors.

d) *True* *False* Two planes in $\mathbb{R}^3$ can meet at a single point.

e) *True* *False* For any two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ we have $\|\vec{u} - 2\vec{v}\| = \|\vec{u}\| + 2\|\vec{v}\|$.

f) *True* *False* The set $\{f$ in C[0, 1] with $\int_0^1 f(x)dx = 1\}$ is a subspace of C[0, 1], where C[0, 1] is the set of continuous real-valued functions on [0, 1].

g) *True* *False* The set of functions $\{1, e^x, \sin(x)\}$ is a basis for C[0, 1].

h) *True* *False* The function $T\colon \mathbb{R}^2 \to \mathbb{R}$, $T(\begin{bmatrix} x \\ y \end{bmatrix}) = 2x - y + 5,$ is a linear transformation.

For Grader's use only:

1			2		3		4			5								TOTAL
a	b	c	a	b	a	b	a	b	c	a	b	c	d	e	f	g	h	